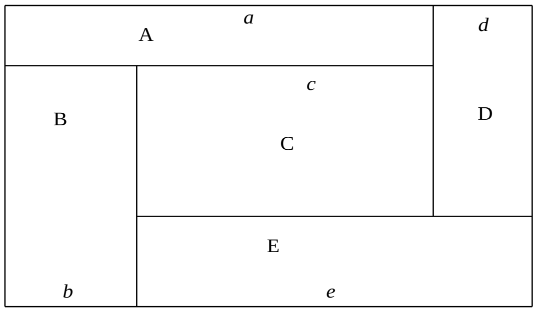
Five Rectangles to Make Another Rectangle

I believe that this problem derives from the February edition of *Scientific American*, where Martin Gardner introduced the problem of finding five rectangles that fit inside another rectangle such that all sides of the rectangles are integers and no two are equal, and the outside rectangle is the smallest possible. My assumption is that no rectangle here is a square.

My memory is not too clear on the details, but it is not clear what "smallest" means in this context. However what follows is one way to address the problem.

The diagram shows the configuration, where rectangles A, B, C, D, and E are placed inside another rectangle. Let the horizontal sides of each be *a*, *b*, *c*, *d*, *e* respectively.



It is not clear whether any of the sides of the inner rectangles may equal one of the sides of the outer rectangle. Initially, we will assume that they may.

We can interpret the question a number of ways, for example:

1. What is the smallest area of the outer rectangle?
2. What is the smallest perimeter of the outer rectangle?
3. What is the smallest central rectangle, C?
4. What is the smallest area and perimeter assuming that no inner side is equal to an outer side?

Consider the 5-tuple (a,b,c,d,e), then we can put certain conditions on their values:

$$a = b+c \text{ and } e = c+d \text{ ....(1)}$$

To prevent finding duplicates differing only in orientation, then take $a > e$ ....(2)

Hence $b > d$, and so $b \geq 2$.

But since all 5 values are positive and different, then $a \geq 5$ ....(3)

But also note that looking at the vertical lines, the situation is exactly analogous, and so the heights of the rectangles form a 5-tuple (a',b',c',d',e') as well. In fact, there are always two ways to make this combination, depending on whether the height of *B* or *D* is taken to be *a'*.

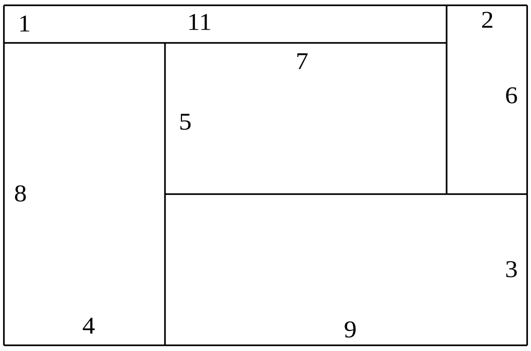
This provides an easy set of criteria to reduce the search space for solutions. We build a table of 5-tuples {*Q*} as follows, and record whenever there is no equality of values.

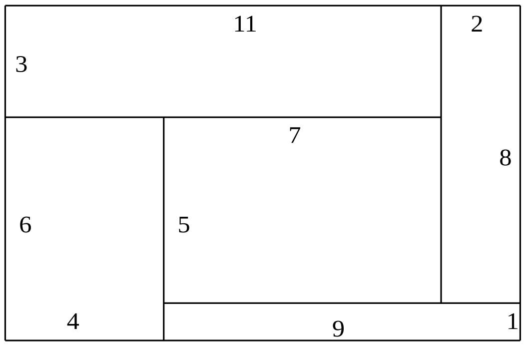
- Search for *a* in 5 : *L*, for some limit *L*.
- Search for *b* in 2 : *a*-1
- Set $c=a-b$
- Search for *d* in 1 : *b*-1
- Set $e=b+c$

- Save the 5-tuple Q, but in doing so, check whether $\exists Q' : Q \cap Q' = \emptyset$, if so record a solution
- From one solution with area, T, we can then set a suitable limit, L, since $6(a+1) \leq T$

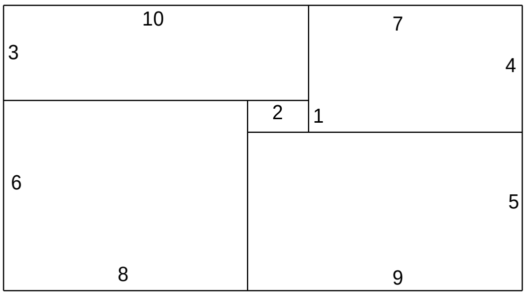
This process is very quick with a computer, and my answers to the 4 questions above are listed below.

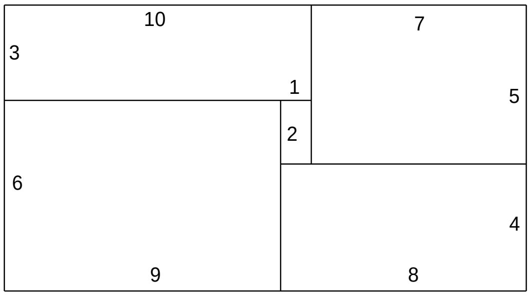
1. The smallest area is found by combining the 5-tuples (11, 4, 7, 2, 9) and (8, 3, 5, 1, 6) to form a rectangle of area 13 x 9 = 117, and perimeter 2(13+9) = 44.

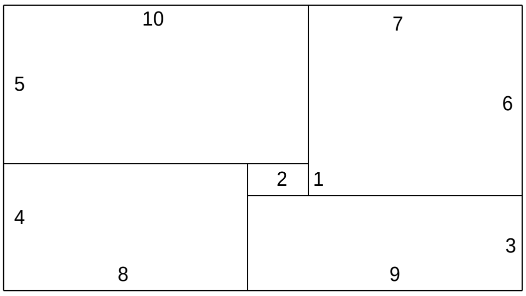


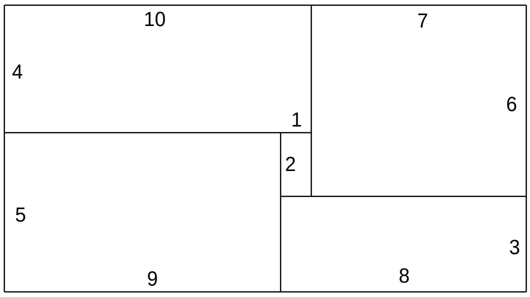


2. The smallest perimeter turns out to be the same combination as the smallest area.
3. The smallest central rectangle is clearly 1 x 2 and can be obtained in two different ways, both having area 153 and perimeter 52.



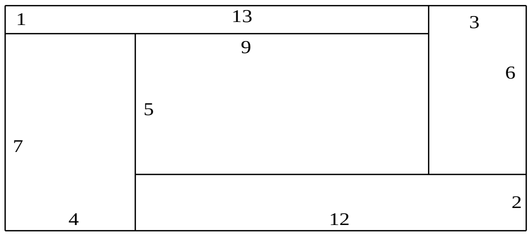


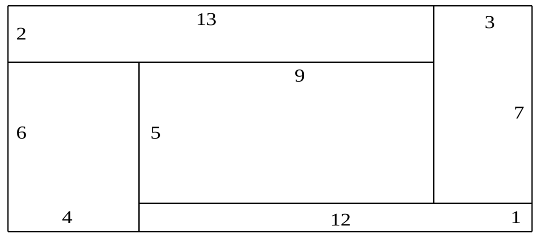




Those at left combine (10, 8, 2, 7, 9) and (6, 5, 1, 3, 4), and at right (10, 9, 1, 7, 8) and (6, 4, 2, 3, 5). In both cases, they use the consecutive numbers 1 to 10 only.

4. In all of the previous examples, one inner side is equal to an outside one, and in every case it is the value 9. For something that does not repeat a side, we can find this:





which combines (13, 4, 9, 3, 12) and (7, 2, 5, 1, 6) to give area of 128 and perimeter 48.

Careful examination of all these results shows that in every case, there are at least two rectangles with equal perimeters. Is it possible to find a division that does not repeat a perimeter? The answer is Yes, and the smallest area of such a rectangle is 126, the smallest perimeter being 50. Can you find it? In this case, only one of the pair of divisions made by a combination fits the conditions, the other having a duplicate perimeter.

Reference

I have a note referring to this column, but have not checked it again.

Scientific American, Feb 1979, “Mathematical Games”, Martin Gardner

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