The Ten Digits

Question

Find a number of ten digits, all digits different, such that the first digit is divisible by 1, the first two as a decimal number is divisible by 2, the first three as a decimal number is divisible by 3, etc. up to the whole ten digits make a number divisible by 10.

So if the number is represented as *abcdefghij* then 2 divides *ab*, 3 divides *abc*, 4 divides *abcd*, etc. up to 9 divides *abcdefghi* and finally *abcdefghij* is divisible by 10. All divisions are without remainder.

Answer

We are given the ten digits as:

a b c d e f g h i j..(0)

Consider division by 1:

a is always divisible by 1, so this adds no extra information.

Consider division by 10:

All multiples of 10 end in zero. Hence *j* = 0, and we now get:

a b c d e f g h i 0..(1)

Consider division by 9:

Since the sum of the nine non-zero digits is 45, which is a multiple of 9, any permutation of the digits will suit. So this condition adds nothing further to the problem.

Consider division by 5:

All multiples of 5 end in 0 or 5. But 0 is taken, so *e* = 5, and we have:

a b c d 5 *f g h i* 0..(2)

Consider division by 2:

Positions 2, 4, 6 and 8 must be divisible by 2, and so only the remaining 4 even digits can go in those 4 places. The other four places must therefore contain the odd digits. Letting *v* represent an even digit, and *o* an odd digit, we can write the required number as:

o v o v 5 *v o v o* 0..(3)

Consider division by 4:

If 4 divides *abcd* then 4 must divide *cd*. But *c* is odd, and so the only combinations that are acceptable here are 12, 16, 32, 36, 72, 76, 92 and 96. So *d* is either 2 or 6. This gives us one of:

o v o 2 5 *v o v o* 0...(4a)

o v o 6 5 *v o v o* 0...(4b)

Consider division by 6:

Note that as *abc* is divisible by 3, then so is abc000, and in fact it is divisible by 6. So we must also have 6 divides *def*. Hence from (4a) and (4b) we can write the desired number as one of:

o v o 2 5 8 *o v o* 0...(5a)

o v o 6 5 4 *o v o* 0...(5b)

Consider division by 8:

We need *fgh* to be divisible by 8. As 8 divides both 400 and 800, we must have 8 divides *gh*. In addition, *b* will be the remaining even digit not yet assigned a value. From each of (5a) and (5b), we get the following options:

o 4 *o* 2 5 8 1 6 *o* 0...(6aa)

o 4 *o* 2 5 8 9 6 *o* 0...(6ab)

o 8 *o* 6 5 4 3 2 *o* 0...(6ba)

o 8 *o* 6 5 4 7 2 *o* 0...(6bb)

Consider division by 3:

There are only three digits left to play with to form possibilities for *abc* in each of the 4 cases.

From (6aa) and (6ab) we quickly find there is no way to get values for *a*4*c* to be divisible by 3 using the available digits of (3, 7, 9) or (1, 3, 7) respectively.

For the others there are several ways. So, also filling in the last remaining digit, we get the options:

1 8 9 6 5 4 3 2 7 0...(7baa)

9 8 1 6 5 4 3 2 7 0...(7bab)

7 8 9 6 5 4 3 2 1 0...(7bac)

9 8 7 6 5 4 3 2 1 0...(7bad)

1 8 9 6 5 4 7 2 3 0...(7bba)

9 8 1 6 5 4 7 2 3 0...(7bbb)

1 8 3 6 5 4 7 2 9 0...(7bbc)

3 8 1 6 5 4 7 2 9 0...(7bbd)

Consider division by 7:

Looking at the 8 possibilities, in only one of them (7bbd) does 7 divide the first 7 digits as a number.

The last possibility is thus:

3 8 1 6 5 4 7 2 9 0...(8)

Of the tests we needed to apply, we have examined the conditions in the order: 1, 10, 9, 5, 2, 4 ,6, 8, 3, 7, which is all of them. So this is the required number, and it is unique:

3 8 1 6 5 4 7 2 9 0...■

Other bases

A natural question to ask now is whether a similar number can be found in other bases. The answer turns out that surprising few small bases will work. These are the answers I've established:

Base **2**: one answer: 1 0

Base **4**: two answers: 1 2 3 0

3 2 1 0

Base **6**: two answers: 1 4 3 2 5 0

5 4 3 2 1 0

Base **8**: three answers: 3 2 5 4 1 6 7 0

5 2 3 4 7 6 1 0

5 6 7 4 3 2 1 0

Base **10**: one answer: 3 8 1 6 5 4 7 2 9 0

Base **14**: one answer: 9 C 3 A 5 4 7 6 B 8 1 2 D 0

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