The Four Digits Problem

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based on notes made 50 years ago.

Make all integers from 0 upwards using the simplest mathematical operators, as defined in some way, and the digits 1, 2, 3 and 4. W. W. Rouse Ball gives this in his book *Mathematical Recreations and Essays*[2] as a more interesting exercise than the normal Four Fours where all the digits are 4.

He states on page 15, that with the digits 1, 2, 3 and 4, and allowing the notation of the denary scale (including decimals), as also algebraic sums, products, and positive integral powers, all numbers from 1 up to 88 can be made. If the use of symbols for square roots and factorials, repeated a finite number of times, is also permitted, then we can get to 264; if negative indices are also permitted, to 276; and if fractional indices are permitted, to 312.

NB: I interpret the last comment about “fractional indices” to mean the extraction of roots other than the square root by saying which one to extract. I cannot find any other solution to 277 and 307 otherwise.

The 89 values I cannot find up to 1000 are: 313, 331, 391, 407, 419, 421, 437, 439, 443, 446, 454, 455, 457, 461, 467, 493, 557, 559, 586, 587, 614, 617, 633, 653, 659, 661, 662, 667, 772, 773, 779, 787, 788, 791, 807, 813, 815, 817, 823, 831, 833, 850, 851, 853, 857, 859, 861, 867, 871, 874, 877, 879, 881, 883, 885, 886, 887, 893, 897, 906, 907, 908, 911, 913, 914, 915, 917, 919, 921, 922, 923, 927, 929, 931, 932, 933, 935, 941, 942, 946, 947, 949, 951, 953, 956, 979, 983, 986, 989.

Examples of each number from 0 upwards are given in the following table. I've tried to keep to the order in which Rouse Ball postulated the use of operators, but also, I've used factorial in preference to the square root sign, and allowed recurring decimals and integral powers in preference to all other powers, factorials and roots.

Some interesting alternative methods are noted, too.

Five digits

Rouse Ball also gives the limits for 5 digits (1, 2, 3, 4 & 5) of 3832 without negative and fractional powers, and of 4282 with. My investigations indicate the same.

THE FOUR DIGITS PROBLEM

No.	Method
0	$1+4-3-2$
1	$3+2-4\times1$
2	$1+2+3-4$
3	$2\times3+1-4$
4	$4\times2-3-1$
5	$4+3-2\times1$
6	$4+3+1-2$
7	$31-24$
8	$4+3+2-1$
9	$4+3+2\times1$
10	$4+3+2+1$
11	$42-31$
12	$4\times3\times(2-1)$
13	$34-21$
14	$21-3-4$
15	$13+4-2$
16	$4\times(3+2-1)$
17	$3\times(1+4)+2$
18	$32-14$
19	$23-4\times1$
20	$21+3-4$
21	$24-3\times1$
22	$43-21$
23	$31-2\times4$
24	$4\times(1+2+3)$
25	$31-4-2$
26	$23+4-1$
27	$23+4\times1$
28	$32-4\times1$
29	$42-13$
30	$13\times2+4$
31	$43-12$
32	$3\times12-4$
33	$34+1-2$
34	$34\times(2-1)$
35	$34+2-1$
36	$32+4\times1$
37	$23+14$
38	$42-3-1$
39	$42-3\times1$
40	$12\times3+4$
41	$43-2\times1$
42	$41+3-2$
43	$43\times(2-1)$
44	$4\times(13-2)$
45	$42+3\times1$
46	$32+14$
47	$41+2\times3$
48	$3\times(14+2)$
49	$41+2^3$
50	$13\times4-2$
51	$12\times4+3$
52	$2\times\left(\frac{3}{\cdot1}-4\right)$ or 4^3-12
53	$3+\frac{2}{\cdot1\times\cdot4}$
54	$4\times13+2$
55	$42+13$
56	$41+\frac{3}{\cdot2}$
57	$3\times\left(\frac{4}{\cdot2}-1\right)$
58	$31\times2-4$
59	$31\times3-4$
60	$4\times(3+12)$
61	$\frac{13}{\cdot2}-4$ or 4^3-2-1

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No.	Method
62	$\frac{31\times4}{2}$
63	$23+\frac{4}{\cdot1}$
64	$23+41$
65	$\frac{13\times2}{\cdot4}$
66	$31\times2+4$
67	$\frac{134}{2}$
68	$34\times2\times1$
69	$3\times(24-1)$
70	$2\times(31+4)$
71	$24\times3-1$
72	$4\times(21-3)$
73	$32+41$
74	$\frac{21}{\cdot3}+4$
75	$3\times(21+4)$
76	$2\times(41-3)$
77	$\frac{(31-\cdot2)}{\cdot4}$
78	$13\times(2+4)$
79	$41\times2-3$
80	$\frac{32\times1}{\cdot4}$ or 3^4-2+1
81	$21\times4-3$
82	$\frac{41}{(\cdot2+\cdot3)}$
83	$\frac{2\times4}{\cdot1}+3$
84	$14\times3\times2$
85	$(41\times2)+3$
86	$43\times2\times1$
87	$(21\times4)+3$

No.	Method
88	$4\times(23-1)$
89	$4!+\frac{13}{\cdot2}$ or $\frac{4!}{\cdot2}-31$
90	$\frac{3\times12}{\cdot4}$
91	$23\times4-1$
92	$23\times4\times1$
93	$23\times4+1$
94	$2\times(41+3!)$
95	$\frac{2+\frac{3}{\cdot4}}{\cdot1}$ or $4!\times(3!-1)-1$
96	$4\times(21+3)$
97	$(1+4)!-23$
98	$(3!-1)!-4!+2$
99	$\frac{3\times4}{\cdot\dot{1}\dot{2}}$ or $123-4!$
100	$\frac{3\times4}{\cdot12}$ or $(4-3)\times\cdot1^{-2}$
101	$3^4+\frac{2}{\cdot1}$ or $\cdot2^{-3}-4!$
102	$34\times(2+1)$
103	$(3!)!\times\cdot2-41$
104	$13\times4\times2$
105	$\frac{42}{\cdot3+\cdot1}$
106	$\frac{3\times4}{\cdot\dot{1}}-2$
107	$\frac{4!}{\cdot2}-13$
108	$4\times(1+2)^3$
109	$\frac{4!-1}{\cdot2}-3!$
110	$\frac{2\times4+3}{\cdot1}$

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No.	Method
111	$(1+4)!-3^2$
112	$4\times\left(\frac{3}{\cdot 1}-2\right)$
113	$\left(\frac{1}{\cdot 2}\right)!-4-3$
114	$3\times\left(\frac{4}{\cdot 1}-2\right)$
115	$23\times(4+1)$
116	$4\times(31-2)$
117	$3\times(41-2)$
118	$\frac{3\times 4}{\cdot 1}-2$
119	$123-4$
120	$\frac{24}{\cdot 3-\cdot 1}$
121	$124-3$
122	$31\times 4-2$
123	$3\times(42-1)$
124	$4\times(32-1)$
125	$41\times 3+2$
126	$42\times 3\times 1$
127	$123+4$
128	$132-4$
129	$3\times(41+2)$
130	$\frac{41-2}{\cdot 3}$ or $124+3!$
131	$\frac{3}{\cdot 2\times\cdot\dot{1}}-4$
132	$134-2$
133	$\frac{\frac{3}{\cdot\dot{1}}-\cdot 4}{\cdot 2}$ or $\frac{4!}{\cdot 2}+13$
134	$\frac{\frac{4}{\cdot 1}+\cdot 2}{\cdot 3}$ or $(3+2)!+14$

No.	Method
135	$\frac{(31-4)}{\cdot 2}$
136	$134+2$
137	$\frac{\frac{3}{\cdot\dot{1}}+\cdot 4}{\cdot 2}$ or $\frac{3}{\cdot 2\times\cdot\dot{1}}+\sqrt{4}$
138	$\frac{\frac{4}{\cdot 3}+2}{\cdot\dot{1}}$ or $(3!-1)!+\frac{4}{\cdot\dot{2}}$
139	$142-3$
140	$\frac{42\times 1}{\cdot 3}$
141	$143-2$
142	$3!\times 4!-2\times 1$
143	$(3\times 4)^2-1$
144	$12\times 4\times 3$
145	$142+3$
146	$\frac{3}{\cdot 1\times\cdot 2}-4$
147	$21\times(3+4)$
148	$\frac{\frac{3}{\cdot 1}-\cdot 4}{\cdot 2}$ $3!\times(4!+1)-2$
149	$(3!)!\times\cdot 2+4+1$
150	$\frac{3\times 2}{\cdot 1\times\cdot 4}$
151	$\frac{31}{\cdot 2}-4$
152	$\frac{\frac{3}{\cdot 1}+\cdot 4}{\cdot 2}$
153	$\frac{31-\cdot 4}{\cdot 2}$
154	$\frac{3}{\cdot 1\times\cdot 2}+4$

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No.	Method
155	$\frac{31\times 2}{\cdot 4}$
156	$132+4!$
157	$\frac{314}{2}$
158	$\frac{\cdot\dot{2}}{\cdot\dot{1}^3}-4$ or $(3!)!\times\cdot 2+14$
159	$\frac{31}{\cdot 2}+4$
160	$32\times(4+1)$
161	$2\times 3^4-1$ or $(3+2)!+41$
162	$1\times 2\times 3^4$
163	$2\times 3^4+1$
164	$2\times(3^4+1)$
165	$\frac{34-1}{\cdot 2}$
166	$\frac{\cdot\dot{2}}{\cdot 1^3}+4$
167	$(3!)!\times\cdot 2+4!-1$
168	$42\times(3+1)$
169	$\frac{34}{\cdot 2}-1$
170	$\frac{34\times 1}{\cdot 2}$
171	$\frac{34}{\cdot 2}+1$
172	$(3!)!\times\cdot\dot{2}+\frac{4}{\sqrt{\cdot\dot{1}}}$
173	13^2+4
174	$4!+\frac{3}{\cdot 1\times\cdot 2}$ or $\frac{(3!)!}{4}-(2+1)!$
175	$\frac{(31+4)}{\cdot 2}$
176	$\frac{\cdot 2}{\cdot 1^3}-4!$

No.	Method
177	$\frac{4}{\cdot 2\times\cdot\dot{1}}-3$
178	$\frac{4!}{\cdot 1\dot{3}}-2$
179	$\frac{31}{\cdot 2}+4!$
180	$\frac{3\times(4+2)}{\cdot 1}$
181	$\frac{(3\times 2)!}{4}+1$
182	$\frac{4!}{\cdot 1\dot{3}}+2$
183	$\frac{4}{\cdot 2\times\cdot\dot{1}}+3$
184	$2^3\times(4!-1)$
185	$\frac{4-\cdot 3}{\cdot 1\times\cdot 2}$
186	$31\times(2+4)$
187	$\frac{4!-3}{\cdot\dot{1}}-2$
188	$(3!)!\times\cdot\dot{1}+\frac{4!}{\cdot\dot{2}}$
189	$\frac{3\times 14}{\cdot\dot{2}}$
190	$\frac{41-3}{\cdot 2}$
191	$4!\times 2^3-1$
192	$4^3\times(2+1)$
193	14^2-3
194	$\frac{43+\cdot\dot{1}}{.\dot{2}}$
195	$\frac{\frac{4}{\cdot\dot{1}}+3}{\cdot 2}$ or $\frac{4!-2}{.\dot{1}}-3$
196	$\frac{\cdot 2}{\cdot 1^3}-4$

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No.	Method
197	$\frac{4}{\cdot 1\times\cdot 2}-3$
198	$\frac{41+3}{\cdot\dot{2}}$
199	14^2+3
200	$\frac{4\times(3+2)}{\cdot 1}$
201	$\frac{4!}{\cdot\dot{1}\dot{2}}+3$
202	$\frac{41}{\cdot 2}-3$
203	$\frac{4}{\cdot 1\times\cdot 2}+3$
204	$\frac{\cdot 2}{\cdot 1^3}+4$
205	$41\times(2+3)$
206	$\frac{23}{\cdot 1}-4!$
207	$\frac{\frac{4}{\cdot 2}+3}{\cdot\dot{1}}$ or $\frac{4!}{\cdot\dot{1}}-3^2$
208	$\frac{41}{\cdot 2}+3$ or $\frac{4!}{\cdot\dot{1}}-2^3$
209	$213-4$
210	$\frac{14\times 3}{\cdot 2}$
211	$214-3$
212	$\frac{4!-3}{\cdot 1}+2$
213	$\frac{24}{\cdot\dot{1}}-3$
214	$\frac{43}{\cdot 2}-1$
215	$\frac{43\times 1}{\cdot 2}$

No.	Method
216	$\frac{24}{\cdot\dot{1}}+3$
217	$213+4$
218	$3!^{(4-1)}+2$
219	$\frac{24}{\cdot\dot{1}}+3$
220	$\frac{41+3}{\cdot 2}$
221	$\frac{4!}{\cdot\dot{1}}+3+2$
222	$\frac{24}{\cdot\dot{1}}+3!$
223	$\frac{4!-2}{\cdot 1}+3$
224	$(4+2)!\times\cdot 3\dot{1}$
225	$(3\times(4+1))^2$
226	$\frac{23}{\cdot 1}-4$
227	$231-4$
228	$3!\times\left(\frac{4}{\cdot 1}-2\right)$ or $\frac{23}{\cdot 1}-\sqrt{4}$
229	$231-\sqrt{4}$
230	$\frac{\frac{4}{\cdot 2}+3}{\cdot 1}$ or $\frac{4!-1}{\cdot 3-\cdot 2}$
231	$\frac{23}{\cdot\dot{1}}+4!$
232	$4\times\left(\frac{3!}{.1}-2\right)$ or $\frac{4!}{\cdot 1}-2^3$
233	$234-1$
234	234×1
235	$234+1$
236	$2\times(3!-1)!-4$
237	$\frac{24}{\cdot 1}-3$

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No.	Method
238	$241-3$
239	$3^{\left(\frac{1}{\cdot 2}\right)}-4$
240	$\frac{4\times 3\times 2}{\cdot 1}$
241	$3^{(4+1)}-2$
242	$243-1$
243	243×1
244	$243+1$
245	$3^{(4+1)}+2$
246	$41\times 3\times 2$
247	$3^{\left(\frac{1}{\cdot 2}\right)}+4$
248	$31\times 4\times 2$
249	$\frac{4!}{\cdot 1}+3^2$
250	$2\times(4+1)^3$
251	$3!\times 42-1$
252	$21\times 4\times 3$
253	$3!\times 42+1$
254	$4^{(3+1)}-2$
255	$231+4!$
256	$4^{(3+2-1)}$
257	$\frac{4!+2}{\cdot 1}-3$
258	$4^{(3+1)}+2$
259	$\left(3!^4-1\right)\times\cdot 2$
260	$\frac{13\times 4}{\cdot 2}$
261	$\frac{4!+3+2}{\cdot\dot{1}}$
262	$2^{\left(4!\times\sqrt{\cdot\dot{1}}\right)}+3!$ or $(3!)!\times\sqrt{\cdot\dot{1}}+4!-2$
263	$\frac{4!}{\cdot 1}+23$
264	$4!\times(13-2)$
265	$\frac{\cdot 1^{-2}+3!}{\cdot 4}$
266	$\frac{3!}{\cdot 2\times\cdot\dot{1}}-4$
267	$3^{\left(\frac{1}{\cdot 2}\right)}+4!$ or $\frac{4!}{\cdot 1}+\frac{3!}{\cdot\dot{2}}$
268	2×134
269	$\frac{(2+3)!}{\cdot\dot{4}}-1$
270	$\frac{23+4}{\cdot 1}$
271	$\frac{(2+3)!}{\cdot\dot{4}}+1$
272	$\frac{4!}{\cdot 1}+32$
273	$\frac{4!}{\cdot 2-\cdot\dot{1}}+3$
274	$\frac{3!}{\cdot 2\times\cdot\dot{1}}+4$
275	$\frac{4!+31}{\cdot 2}$
276	$(4+1)!\times(2+\cdot 3)$ or $\frac{23\times 4}{\sqrt{\cdot\dot{1}}}$
277	$34+\sqrt[(-\cdot 2)]{\sqrt{\cdot\dot{1}}}$
278	$(3!)!\times\cdot 4-\sqrt{\cdot 1^{-2}}$ or $\sqrt{\sqrt{\sqrt{\cdot 1^{-4!}}}}-(3!)!-2$
279	$\frac{3+\cdot\dot{4}}{\cdot\dot{1}^2}$
280	$\frac{21\times 4}{\cdot 3}$
281	$\frac{\cdot 2^{-3}-\cdot\dot{1}}{\cdot\dot{4}}$
282	$4!\times 12-3!$

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No.	Method
283	$(3!)!\times\cdot 4-\frac{1}{\cdot 2}$
284	$\frac{32}{\cdot\dot{1}}-4$
285	$4!\times 12-3$
286	143×2
287	$\frac{4^3}{\cdot\dot{2}}-1$
288	$\frac{34-2}{\cdot\dot{1}}$
289	$(13+4)^2$
290	$\frac{4!+3+2}{\cdot 1}$
291	$4!\times 12+3$
292	$\frac{32}{\cdot\dot{1}}+4$
293	$(3!)!\times\cdot 4+\frac{1}{\cdot 2}$
294	$42\times(3!+1)$
295	$(3!)!\times\cdot 41-\cdot 2$
296	$\frac{3}{\cdot 1^2}-4$
297	$\frac{132}{\cdot\dot{4}}$ or $321-4!$
298	$\frac{\frac{3!}{\cdot 1}-\cdot 4}{\cdot 2}$ or $\frac{4!+3!}{\cdot 1}-2$
299	$\frac{(3+2)!}{\cdot 4}-1$
300	$\frac{3}{\cdot 1^{(4-2)}}$ or $(3!)!\times\cdot 4+12$
301	$\frac{(3+2)!}{\cdot 4}+1$
302	$\frac{\frac{3!}{\cdot 1}+\cdot 4}{\cdot 2}$ or $\frac{4!+3!}{\cdot 1}+2$
303	$\frac{\left(\frac{1}{\cdot 2}\right)!}{\cdot 4}+3$ or $\frac{\cdot 2^{-3}-4!}{\sqrt{\cdot\dot{1}}}$
304	$\frac{34}{\cdot\dot{1}}-2$ or $\frac{3!}{\cdot 2\times\cdot 1}+4$
305	$\frac{(3!-1)!+2}{\cdot 4}$
306	$\frac{34}{\cdot\dot{2}-\cdot\dot{1}}$ or $\frac{32+\sqrt{4}}{\cdot\dot{1}}$
307	$\cdot 3\times\sqrt[(\cdot 1)]{\sqrt{4}}-\cdot 2$
308	$312-4$
309	$(3!)!\times\cdot\dot{4}+21$
310	$4!\times 13-2$
311	$(3!)!\times\cdot\dot{4}-\sqrt{\cdot\dot{1}^{-2}}$
312	$314-2$

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Notation

The midpoint (·) is used for the decimal point.

Recurring decimals are written with dots over the first and last digit of the recurring sequence. So $\cdot\dot{1}$ is the same as $\cdot 1111\ldots$ (i.e. one ninth), $\cdot\dot{1}\dot{2}$ is the same as $\cdot 121212\ldots$ (i.e. twelve ninety-ninths, or four thirty-thirds), and $\cdot 1\dot{3}$ is the same as $\cdot 133333\ldots$ (i.e. twelve ninetieths, or two fifteenths). Some care is required to see these dots and some printers may not be able to print them well.

Addition, subtraction and multiplication are as normal, but division is always given as a fraction.

The factorial function is represented as usual by a ! giving $3! = 6$ and $4! = 24$.

Square roots are also as usual, with $\sqrt{4} = 2$ and $\sqrt{\cdot\dot{1}} = 1/3$.

To get 277 and 307, and some higher numbers, I have assumed that Rouse Ball was using the notation of an n-th root where n is not integral. So $\sqrt[\cdot 1]{2} = 2^{10} = 1024$, and negative roots are the equivalent of reciprocal powers, so $\sqrt[-\cdot 2]{\cdot\dot{3}} = (1/3)^{-5} = 3^{5} = 243$.

References

[1] Creative Commons License, *http://creativecommons.org/licenses/by/2.5/*

[2] W.W. Rouse Ball, revised by H.S.M. Coxeter, *Mathematical Recreations and Essays, 11th edition*, 1939, reprinted in 1967 by Macmillan, p15ff.