Pythagorean Quadrilaterals

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This is a modified version of a paper first published as [AP].

In Figure 1, we have a convex quadrilateral whose diagonals AB, CD, intersect at right angles at O). The question is to find four right triangles with integer-valued sides which fit together to form the quadrilateral.

In particular, is it possible for the triangles all to be primitive and no two similar?

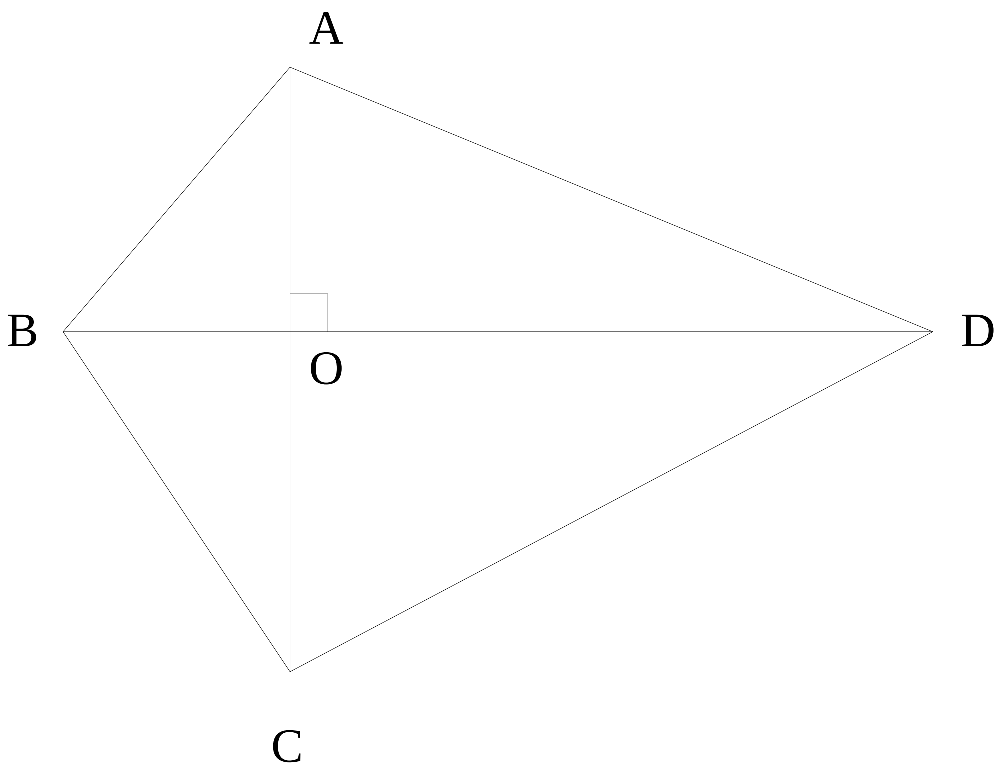

Figure 1: Convex quadrilateral with diagonals at right angles

Almost any number theory text shows that the complete solution in integers of the Pythagorean relation $x^2 + y^2 = z^2$ is given by

$$\left.\begin{array}{rcl} x & = & (p^2-q^2)r \\ y & = & 2pqr \\ z & = & (p^2+q^2)r \end{array}\right\} \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(S)$$

where $(p, q) = 1$, and one of p, q is even. If $r = 1$, then the triangle is primitive.

We need to find four solutions (p_A, q_A, r_A) to S, A = 1, 2, 3, 4 such that

$$
\begin{aligned}
p_1 q_1 r_1 &= p_2 q_2 r_2 \\
p_3 q_3 r_3 &= p_4 q_4 r_4 \\
(p_1 - q_1)(p_1 + q_1) r_1 &= (p_4 - q_4)(p_4 + q_4) r_4 \\
(p_2 - q_2)(p_2 + q_2) r_2 &= (p_3 - q_3)(p_3 + q_3) r_3
\end{aligned}
$$

Mr. H. ApSimon [HA] found one solution (without a computer) to this set, which is the one with the lowest possible values:

(3, 2, 5) (10, 3, 1) (4, 3, 13) (13, 12, 1)

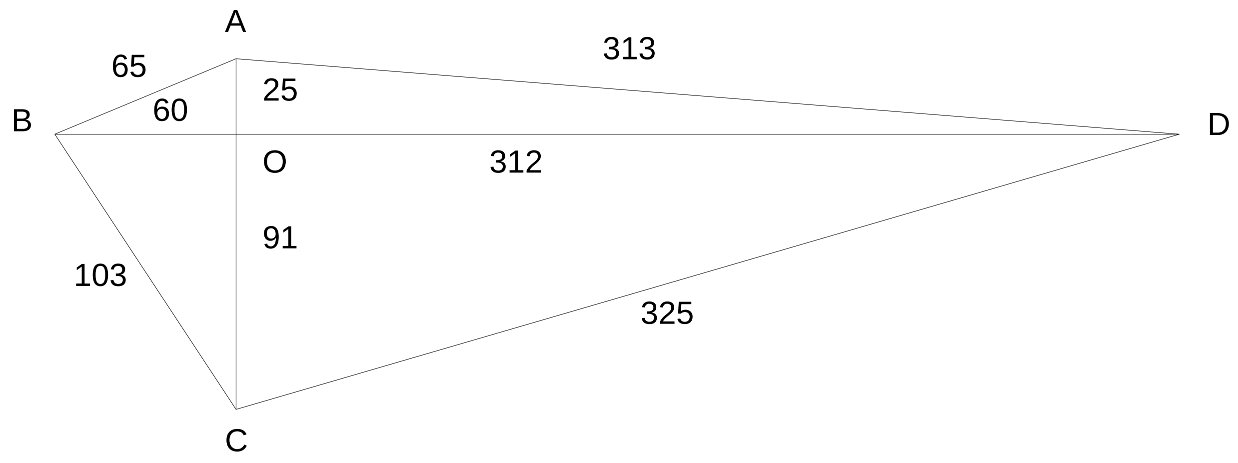

Figure 2: The smallest Pythagorean quadrilateral

This has two primitive triangles. The sides of the triangles are:

(25, 60, 65)	(91, 60, 109)	(91, 312, 325)	(25, 312, 313)
ΔBOA	ΔCOB	ΔCOD	ΔDOA

Figure 2 illustrates this with the corresponding triangles indicated below the values shown above. A computer search performed in 1987 quickly turned up many more.

I then restricted the search to just those solutions with three primitive triangles and was surprised to find, relatively soon, a solution with all four triangles primitive. The complete list where the shortest odd leg is less than 10,000,000 is given in Table 1. The solution, shown in Figure 3, with the smallest values has sides of the triangles:

(28453, 20292, 34933) (284795, 20292, 285517)

(28435, 181908, 184117) (284795, 181908, 337933)

Table 1: Solutions of (S), where r = 1

p_1	q_1	p_2	q_2	p_3	q_3	p_4	q_4
178	57	534	19	558	163	326	279

p_1	q_1	p_2	q_2	p_3	q_3	p_4	q_4
388	323	1292	97	2024	1561	1784	1771
1525	1484	3233	700	3175	344	1075	1016
739	610	3695	122	3745	622	1555	1498
3175	3116	4100	2413	3320	187	913	680
1697	1418	3394	709	3886	2021	2881	2726
1027	430	5135	86	5215	914	2285	2086
2215	1886	9430	443	9710	2357	4714	4855
3454	3121	6242	1727	6554	2641	4294	4031
1588	531	3573	236	7347	6424	6952	6789
2363	1414	28078	119	50422	41881	45973	45934
16022	15909	24033	10606	21567	194	2522	1659
10730	10509	16385	6882	15295	3582	7562	7245
2498	1127	61201	46	61537	6422	19942	19817
2924	1841	4471	1204	4577	1552	3184	2231
3101	2028	4732	1329	5488	3081	4459	3792
6986	6563	45941	998	45931	278	3994	3197
2679	764	26931	76	27237	4072	10689	10376
7610	7163	72295	754	86903	48230	64766	64715
12033	11744	55008	2569	54948	49	2793	964
55366	55301	61807	49538	36961	154	3214	1771
5545	4838	90938	295	92062	14345	36391	36290
39268	39173	422131	3644	422117	1208	22664	22499
5698	4841	7238	3811	6542	2221	4442	3271

Note: These figures agree with those found by Randy Dougherty [RD] for the larger odd side up to 21,051,829

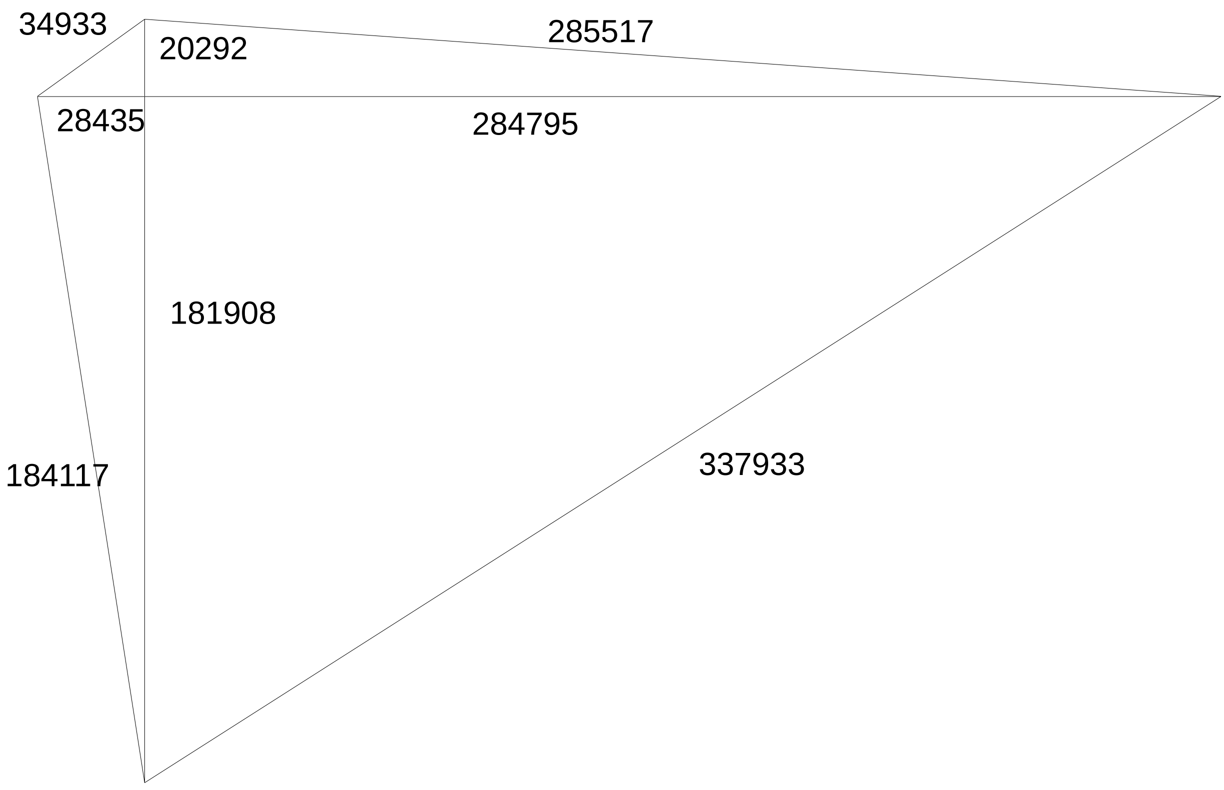

Figure 3: The smallest primitive Pythagorean quadrilateral

I then turned my attention to an *opposite* question. What is the smallest solution where none of the *r*'s are equal to 1, but where $(r_a, r_b) = 1$ for all pairs (a, b), $a \neq b$, i.e., where none of the triangles are primitive, but no two triangles are similar. Table 2, lists the first twenty found. The smallest solution (the second row in the table), shown in Figure 4, in this case has sides of:

(1395, 2200, 2605) (6699, 2200, 7051)

(1395, 868, 1643) (6699, 868, 6755)

Table 2: Solutions of (S), where all r's are mutually co-prime

p_1	q_1	r_1	p_2	q_2	r_2	p_3	q_3	r_3	p_4	q_4	r_4
17	14	13	26	7	17	31	20	19	20	19	31
20	11	5	25	4	11	31	2	7	7	2	31
37	30	5	25	6	37	28	9	31	36	31	7
47	40	5	25	8	47	49	2	11	22	7	7
52	47	7	91	4	47	143	2	19	26	19	11
18	13	23	207	2	13	174	31	19	58	57	31

p_1	q_1	r_1	p_2	q_2	r_2	p_3	q_3	r_3	p_4	q_4	r_4
28	5	5	25	4	7	26	23	29	29	26	23
28	19	11	44	19	7	47	2	5	10	1	47
49	38	5	49	10	19	58	1	13	13	2	29
20	17	55	44	25	17	37	14	19	19	14	37
19	18	203	63	58	19	37	18	11	18	11	37
187	184	7	56	17	253	106	37	73	74	73	53
913	912	5	16	15	17347	146	63	31	63	62	73
26	11	17	34	11	13	37	28	23	28	23	37
74	59	5	118	5	37	112	1	41	41	16	7
69	68	77	132	119	23	137	114	13	39	38	137
22	21	247	114	91	11	129	124	41	124	123	43
22	21	253	138	121	7	43	6	17	17	6	43
71	52	5	20	13	71	52	19	7	28	13	19
24	5	23	69	8	5	58	3	7	21	2	29

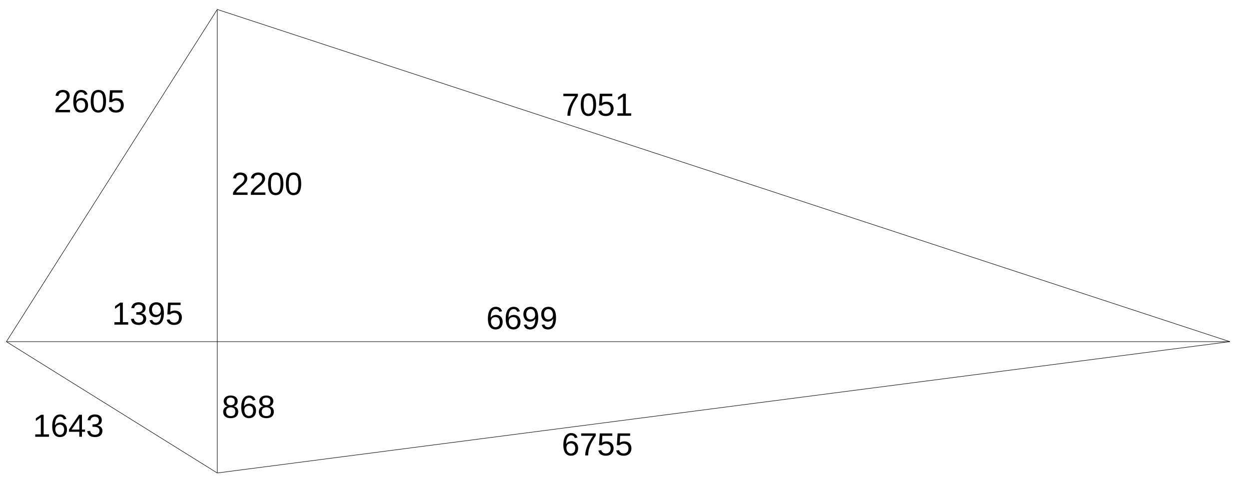

Figure 4: The smallest anti-primitive Pythagorean quadrilateral

Historical Note

This work was done in a fairly slack period of work. Hugh ApSimon used a different way of presenting the results that make it clear what the derivations are, but can generate large numbers fairly quickly when the search is left running. The machines in those days were rather slow, but the problem is now much more approachable with the speed of modern machines at our disposal. It needs multiple length arithmetic fairly soon, and makes a good problem for teaching.

References

[AP] A. Pepperdine, *Journal of Recreational Mathematics*, vol 21(1), 1989

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[HA] H. ApSimon, *Private correspondence to A. Pepperdine,* 1988-02-13

[RD] R. Dougherty, *Private correspondence to H. ApSimon,* 1983-12-12

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