Sets of Integers with Unique Pair-wise Sums

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In Neil Sloane's excellent resource[2], sequence A010672 has the property that each additional term is the minimum one that ensures that all possible pairwise sums of distinct terms in the sequence so far are distinct. Duplicate terms are not allowed, nor are pairs allowed to be the same term taken twice. It starts like this

0, 1, 2, 4, 7, 12, 20, 29, 38, 52, 73, 94, 127, ...

If the terms may be repeated in a pair, the sequence is known as the Mian-Chowla sequence[3].

Commentary

Instead of a continuing sequence, suppose one looks at sets of non-negative integers of increasing size in which each set has the property that all pairwise sums of elements are distinct. Then we can ask what the minimum value is for the maximum element in each set. Or we can ask what the minimum value is for the sum of all elements in each set.

The sets so defined are not the same as initial sequences of A010672 in [2], nor are those with the minimum highest element the same as those with the smallest sum. In general they are not easy to prove rigorously as optimum in each case.

Solution

The following tables shows the results obtained by a simple exhaustive computer search for low cardinality of sets. For a given cardinality, there may be more than one set with the optimum sum or highest element.

The sets with the minimum largest element are shown in Table 1. From each set another equivalent one (not listed) can be made by subtracting each element from the largest element.

# elem	Elements in set										
2	0	1									
3	0	1	2								
4	0	1	2	4							
5	0	1	2	4	7						
6	0	1	2	4	7	12					
	0	1	2	6	9	12					
7	0	1	2	4	8	13	18				
	0	1	2	7	10	14	18				
8	0	1	2	4	8	14	19	24			

# elem	Elements in set										
9	0	1	2	4	8	15	24	29	34		
	0	1	2	7	13	16	26	30	34		
	0	1	2	14	19	24	27	30	34		
10	0	1	7	10	13	21	26	41	43	45	
11	0	1	5	9	17	31	33	44	51	54	57
	0	1	5	9	17	31	34	37	44	55	57

Table 1: Sets with minimum largest non-negative element

It is even harder to find the sets with minimum sum of elements. In this case, there is no equivalent, and all sets with the smallest sum are given for a given cardinality; but an exhaustive search found the solutions shown in Table 2.

# elem	Elements in set											Sum
2	0	1										1
3	0	1	2									3
4	0	1	2	4								7
5	0	1	2	4	7							14
6	0	1	2	4	7	12						26
7	0	1	2	4	7	12	20					46
	0	1	2	4	8	13	18					
8	0	1	2	4	8	14	19	24				72
9	0	1	2	4	8	14	19	24	40			112
10	0	1	2	4	8	14	19	24	40	49		161
11	0	1	2	4	8	15	24	29	34	46	64	227

Table 2: Sets with smallest sum of non-negative elements

References

[1] Creative Commons License, *http://creativecommons.org/licenses/by/2.5/*

[2] N.J.A. Sloane, "The Online Encyclopedia of Integer Sequences", http://www.research.att.com/~njas/sequences/

[3] Mian-Chowla Sequence, http://en.wikipedia.org/wiki/Mian-Chowla_sequence

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