Sums of Powers

This note is about expressing the sum of the (n+1)-th powers of n variables in terms of the sums of lower powers.

Initial comments

To start with, we can note that

$$\begin{aligned}(a+b)^3 &= a^3+3a^2b+3ab^2+b^3 \\ &= 3a^3+3a^2b+3ab^2+3b^3 - 2(a^3+b^3) \\ &= 3(a+b)(a^2+b^2) - 2(a^3+b^3)\end{aligned}$$

If we are to define

$$s_n = a^n+b^n$$

then we can write the relation as

$$s_1^3 = 3s_1s_2 - 2s_3 \text{ ....(1)}$$

Or, to write it in a form that expresses larger powers in terms of smaller ones:

$$2s_3 = -s_1^3 + 3s_1s_2 \text{ ....(2)}$$

Alternative derivation for cubic relation

The above just popped out by inspection of a cubic expression of two variables. Can we derive it in a more systematic way, perhaps one which can us a method to apply to higher powers?

Looking at (1) it is clear that each term has degree 3. So let's consider all the ways in which a term of degree 3 may be made up. They are of the forms a^3, or a^2b, and other terms that are equivalent to those two by permuting or selecting a combination of the two terms a and b. In all terms, the sum of the indices must be 3.

Suppose that such a formula could be found, then we could write it is as

$$s_3 = ps_1^3 + qs_1s_2 \text{ ....(3)}$$

where we have taken the coefficient of the s_3 term to be unity.

Let us now consider expansions of each of these terms. We are looking for an algebraic identity, and so we have to equate the coefficients of each type of term in turn.

Term a^3. There is just one way of selecting a from each possible factor in each of the terms to make up a^3, and so

$$1 = p + q \text{ ....(4)}$$

Term a^2b. Here, s_3 does not contain a term of this type. There are three ways of selecting two factors containing a and one of b in s_1^3, and only a single way of getting it from the last term. So we have

$$0 = 3p + q \text{ ....(5)}$$

So there are two equations in 2 unknowns. From (5), we can write

$$q = -3p$$

Substituting that in (4) gives us

$$1 = p - 3p = -2p$$

Hence, $p = -1/2$ and $q = 3/2$, giving the result;

$$s_3 = \frac{-s_1^3}{2} + \frac{3s_1 s_2}{2}$$

Multiplying through by 2 gives us (2) as required.

At this point we can also see why we cannot find an equivalent formula for three variables. If there were, then one of the terms would be like *abc*. The only way of getting such a term would be from the s_1^3 term, which would necessitate *p* being zero, and then it is clear that the conditions on the other terms cannot be met.

Higher powers

Let's see whether we can do the same thing for fourth powers, but in this case, we can have three variables. We want to find a relation of the form

$$ps_1^4 + qs_1^2 s_2 + rs_1 s_3 + ts_2^2 + us_4 = 0$$

We will be equating coefficients of the terms containing up to 3 variables whose powers sum to 4, and since the situation is symmetric in all the variables, we can consider them in non-increasing order of powers from *a* to *b* to *c*.

The terms to consider will be of the forms

$$a^4,\ a^3b,\ a^2b^2, \text{ and } a^2bc$$

Let's start by looking at the s_1^4 term. This can be written as

$$s_1^4 = (a+b+c)^4 = (a+b+c)(a+b+c)(a+b+c)(a+b+c)$$

The expansion will contain the term a^4 just once, since we must select one *a* from each bracket, and there is only one way of doing that.

But there are 4 ways of making a^3b, since we can select the *b* from just one bracket in four ways, and then the other three must use the *a* term.

There are 6 ways of making a^2b^2, since we have to select two sets of brackets from the 4 to contain *a*, and then the other two must contain a *b*. The notation for representing the number of ways of selecting *k* things from a total of *n*, where the order is not relevant has varied greatly over the years, and you may see any of the following

$${}_nC_k = {}^nC_k = C_k^n = C(n,k) = \binom{n}{k}$$

The last one is now common, and the one I will use here. Its computation is straightforward.

$$\binom{n}{k} = \frac{n!}{k!(n-k)!} = \frac{n\times(n-1)\times\ldots\times(n-k+1)}{1\times2\times\ldots\times k}$$

In our case, this is

$$\binom{4}{2} = \frac{4!}{2!2!} = \frac{4\times3}{1\times2} = 6$$

The last one is a^2bc . Here we need to take two brackets containing and *a*, and then the other two must provide a *b* and a *c*. There are as before 6 ways of selecting the two brackets for the *a*'s, but to provide the other two, there are two ways of doing it, so the answer is 12.

That completes the first column in the table below.

The next thing to consider is the term

$$s_1^2 s_2 = (a^2+b^2+c^2)(a+b+c)(a+b+c)$$

where looking at how it provides each of the terms, as before, provides the second column.

So each of the three terms can supply a power of *a* in just one way, so there is one way only of providing a^4 .

For a^3b , the first bracket can provide only the *a* in one way, and then it supplies two powers. The last can come fom either of the other two brackets. So there are two ways of making this term.

For a^2b^2 , the first brackets provides either two *a*'s or two *b*'s, and the other two brackets must then supply the other parts of the term. Hence there are two ways of getting the result.

And finally, the term part must come from the first bracket. So there are just two ways of it supplying a contribution.

That completes the second column.

The other columns are easy enough, and are left for you to satisfy yourself that the table is correct.

We can now build a table for each coefficient.

	s_1^4	$s_1^2 s_2$	$s_1 s_3$	s_2^2	s_4
	p	q	r	t	u
a^4	1	1	1	1	1
a^3b	4	2	1	0	0
a^2b^2	$\binom{4}{2}=6$	2	0	2	0
a^2bc	$\binom{4}{2}\times2=6$	2	0	0	0

The largest value in the table is 12, so if we set *p* = 1, then we get the simultaneous equations:

$$1 + q + r + t + u = 0 \quad\text{...(6)}$$

$$4 + 2q + r = 0 \quad\text{...(7)}$$

$$6 + 2q + 2t = 0 \quad\text{...(8)}$$

$$12 + 2q = 0 \quad\text{..(9)}$$

These can be easily solved by working up from the (9):

$$q = -6 \quad \text{...(10)}$$

Substituting (10) in (8)

$$t = -3-q = -3+6 = 3 \quad \text{..(11)}$$

Putting (10) in (7)

$$r = -4-2q = -4+12 = 8 \quad \text{..(12)}$$

And finally using (10), (11) and (12) in (6) gives

$$u = -1-q-r-t = -1+6-8-3 = -6 \quad \text{................................(13)}$$

Hence the identity is:

$$s_1^4 - 6s_1^2s_2 + 8s_1s_3 + 3s_2^2 - 6s_4 = 0$$

Or, as is often written:

$$6s_4 = s_1^4 - 6s_1^2s_2 + 8s_1s_3 + 3s_2^2 \quad \text{..} \blacksquare$$

Going higher

For fifth powers, we have to be even more careful while counting the ways in which each term is to be found in each expansion. The possible ways of making fifth powers from combinations of sums of other powers. We can, except for s_5, match each one with a corresponding type of term, and so making the correct number of simultaneous equations to be solved for the coefficients in the resulting expression.

p	q	r	t	u	v	1
s_5	$s_4 s_1$	$s_3 s_2$	$s_3 s_1^2$	$s_2^2 s_1$	$s_2 s_1^3$	s_1^5
a^5	$a^4 b$	$a^3 b^2$	$a^3 bc$	$a^2 b^2 c$	$a^2 bcd$	

The correspondence can now be seen. Each suffix in row 2 matches a power in row 3, and each power in row 2 matches multiple variables in row 3. The same will be true for each higher power.

Let’s find an expression of the form:

$$s_1^5 = ps_5 + qs_4s_1 + rs_3s_2 + ts_3s_1^2 + us_2^2s_1 + vs_2s_1^3 \quad \text{.....................(14)}$$

We consider expanding each term from row 2 but we need only consider those terms where the indices do not decrease from variable to variable, in other words only those listed in row 3.

Examining the table from right to left:

$$s_1^5 = (a+b+c+d)^5 = \begin{aligned}&a^5\\ +\;&5a^4(b+\ldots)\\ +\;&10a^3(b^2+\ldots)\\ +\;&10a^3(2bc+\ldots)\\ +\;&10a^2(3b^2c+\ldots)\\ +\;&10a^2(6bcd\ldots)\\ +\;&\ldots\end{aligned}$$

$$s_2 s_1^3 = (a^2+b^2+c^2+d^2)(a+b+c+d)^3 = \begin{aligned}&a^5\\ +\;&3a^4(b+\ldots)\\ +\;&3a^3(b^2+\ldots) + a^3(b^2+\ldots)\\ +\;&3a^3(2bc+\ldots)\\ +\;&a^2(3b^2c+\ldots) + 3a^2(b^2c+\ldots)\\ +\;&a^2(6bcd\ldots)\\ +\;&\ldots\end{aligned}$$

$$s_2^2 s_1 = (a^2+b^2+c^2+d^2)^2(a+b+c+d) = \begin{aligned}&a^5\\ +\;&a^4(b+\ldots)\\ +\;&2a^3(b^2+\ldots)\\ +\;&2a^2(b^2c+\ldots)\\ +\;&\ldots\end{aligned}$$

$$s_3 s_1^2 = (a^3+b^3+c^3+d^3)(a+b+c+d)^2 = \begin{aligned}&a^5\\ +\;&2a^4(b+\ldots)\\ +\;&a^3(b^2+\ldots)\\ +\;&a^3(2bc+\ldots)\\ +\;&\ldots\end{aligned}$$

$$s_3 s_2 = (a^3+b^3+c^3+d^3)(a^2+b^2+c^2+d^2) = \begin{aligned}&a^5\\ +\;&a^3(b^2+\ldots)\\ +\;&a^2(b^3+\ldots)\\ +\;&\ldots\end{aligned}$$

$$s_4 s_1 = (a^4+b^4+c^4+d^4)(a+b+c+d) = \begin{aligned}&a^5\\ +\;&a^4(b+\ldots)\\ +\;&\ldots\end{aligned}$$

$$s_5 = a^5+\ldots$$

From which we get when we compare equivalent terms in both sides of (14) the 6 simultaneous equations:

$$\begin{array}{rcccccccccccccc}
a^5: & p & + & q & + & r & + & t & + & u & + & v & = & 1\\
a^4b: & & & q & & & + & 2t & + & u & + & 3v & = & 5\\
a^3b^2: & & & & & r & + & t & + & 2u & + & 4v & = & 10\\
a^3bc: & & & & & & & 2t & & & + & 6v & = & 20\\
a^2b^2c: & & & & & & & & & 2u & + & 6v & = & 30\\
a^2bcd: & & & & & & & & & & & 6v & = & 60
\end{array}$$

This is already in a suitable diagonal form for the solution to be found from the bottom up:

$$v = 10,\ u = -15,\ t = -20,\ r = 20,\ q = 30,\ p = -24 \quad\text{.............................(15)}$$

hence the appropriate expression for fifth powers is:

$$s_1^5 = -24 s_5 + 30 s_4 s_1 + 20 s_3 s_2 - 20 s_3 s_1^2 - 15 s_2^2 s_1 + 10 s_2 s_1^3 \ \text{................}\blacksquare$$

Alternatively

There is another way of finding these coefficients. But from a mathematical perspective, it is not so attractive. First we ought to establish that such a relation does in fact exist. The constructive techniques above do that, but what comes below assumes such is the case.

Because we are looking for an identity, we could compute the terms for 6 independent quadruples, and so set up 6 simultaneous equations, which can be solved in the usual way. For instance, from the sets

{0, 0, 0, 1}, {0, 0, 1, 1}, {0, 0, 1, 2}, {0, 1, 1, 2}, {0, 1, 2, 2} and {1, 1, 2, 2}

we get the equations (using the same coefficients from (14)):

$$\begin{pmatrix} 1 & 1 & 1 & 1 & 1 & 1 \\ 2 & 4 & 4 & 8 & 8 & 16 \\ 33 & 51 & 45 & 81 & 75 & 135 \\ 34 & 72 & 60 & 160 & 144 & 384 \\ 65 & 165 & 153 & 425 & 405 & 1125 \\ 66 & 204 & 180 & 648 & 600 & 2160 \end{pmatrix} \begin{pmatrix} p \\ q \\ r \\ t \\ u \\ v \end{pmatrix} = \begin{pmatrix} 1 \\ 32 \\ 243 \\ 1024 \\ 3125 \\ 7776 \end{pmatrix}$$

Solving with any standard technique will give the same answers as in (15).

The advantage of this technique is that it does not rely on you being able to accurately count the terms in all of the expansions, and thus would be a preferable method when considering powers higher than 5. For 6-th powers, for example, we need 10 equations and the chances of making a mistake are correspondingly higher. A spreadsheet can be used to calculate, and help solve these more straightforward ones.

Andy Pepperdine

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