Dissecting a Triangle into Isosceles Triangles

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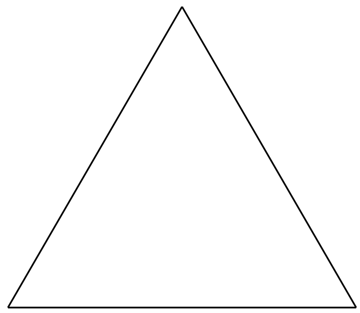


The task is to dissect an equilateral triangle with straight cuts into a number of pieces each of which is an isosceles triangle such that no piece is an equilateral triangle, and no two triangles are similar, and to do so in the fewest pieces. How many pieces do we need?

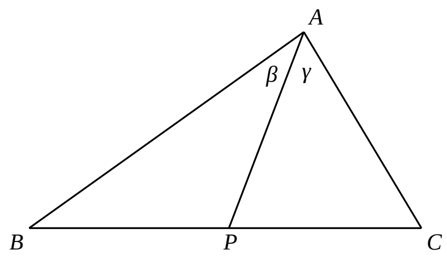
Sad to say, I have lost both the original statement of the problem and where it came from that caused this investigation about 30 years ago, at a guess.

We will start by looking at general triangles until we can form something that can satisfy the question.

1 Piece

Type 1. First, if there is only one piece, then it is the full triangle, and so that can only happen when the triangle is isosceles, including when it is equilateral. But then it does not satisfy all the condition required in the statement of the problem.

2 Pieces



What sort of triangles can we dissect into two isosceles pieces? Clearly the only way to make two triangles is to cut from one vertex to cross the opposite side. What sort of conditions do we have when we do that?

Assume that *AP* makes angles as shown, and that ∠APB is obtuse. Then clearly, the triangle APB can be isosceles only when AP = PB.

In that case, $\beta = B$, and $A > B$. But the triangle *APC* can be isosceles in three possible ways.

Type 2a. If $PA = PC$, then *P* is the circumcentre of triangle *ABC*, and hence *BC* is a diameter, and the angle at *A* is a right angle. Any right-angled triangle can be cut this way into two isosceles triangles.

Type 2b. If $CA = CP$, then $\angle APC = \gamma = \beta + B = 2B$. Hence $A = \beta + \gamma = 3B$. In this case, from triangle *ABC*, $A+B = 4B = 180º - C < 180º$. Hence $B < 45º$, for this dissection to be possible.

Note that if $A = 60º$, then $B = 20º$ and $C = 100º$.

Type 2c. If $AP = AC$, then $\angle APC = C = B + \beta = 2B$, and so $A = 180º - B - C = 180º - 3B > 0$, and so for this case, the condition $B < 60º$ must hold.

Note that if $A = 60º$, then $B = 40º$ and $C = 80º$.

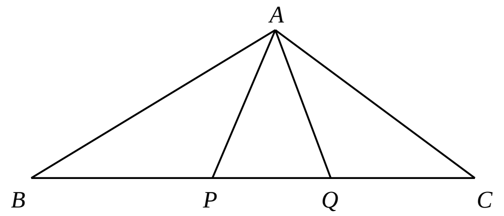
That exhausts the ways of dissecting a triangle into two isosceles triangles.

3 Pieces

Looking at the type 2 cases, we see that they are formed by adding an isosceles triangle onto a side of another triangle, by adding *APB* onto side *AP* of triangle *APC*. This might be done in 6 ways, since there three sides, and each side might be extended at either end by an amount equal to the adjacent side. E.g. extend *CP* by *PB* = *PA*. If the original triangle was also isosceles, then in practice there would be only three possible extensions due to symmetry of the first triangle.

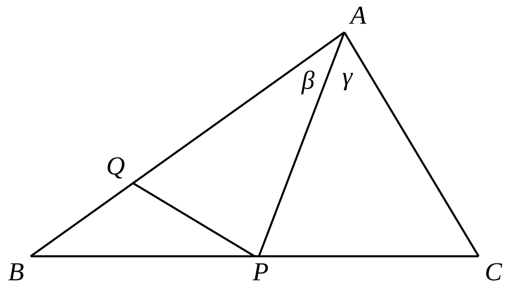
These can be represented as follows.

Type 3a. Two lines drawn through the same vertex to intersect the opposite side. There are two possibilities for the central triangle to be isosceles, either *AP* = *AQ*, or *AP* = *PQ*. Obviously, in both cases, angles *B* and *C* cannot both be 60º. In fact neither can, since they are not divided by a line drawn through them.



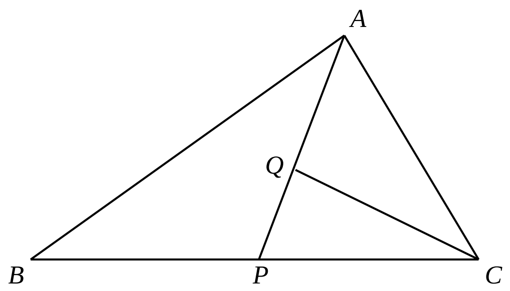
I'll leave it as an exercise for the reader to work out the conditions such that the triangle *ABC* can be dissected in this manner.

Type 3b. One line drawn through *A* to meet *BC* in *P*, and then another through *P* to meet *AB* in *Q*. In this case, if *ABC* is equilateral, then *C* is 60º, but that means that whether *AP* = *AC*, or *PA* = *PC*, in both cases all angles of triangle *APC* are 60º, making it equilateral, and so not satisfying the conditions of the problem.



Again, if you are interested, then I'll let you work out the conditions for the shape of triangle *ABC* for this dissection to be applicable.

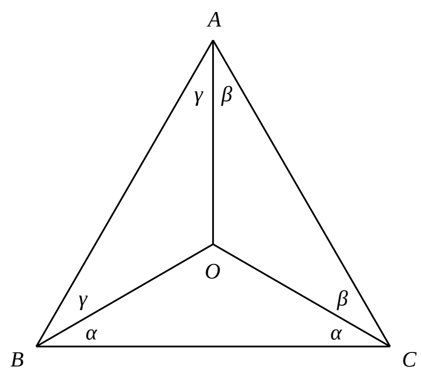
Type 3c. A line through *A* and another through *C*. But this fails for our problem in a similar way to tyupe £b, since the reasoning applied to APC there applies to APB here.



It must now be clear we have wasted a lot of time because the arguments against these types of dissetion all fail to divide all three of the angles of the equilateral triangle into two other angles. We must cut through all of the angles *A*, *B* and *C*.

How do we accomplished that and still produce just three pieces? There is a way, and it makes all three triangles to be isosceles so long as the original triangle is strictly acute.

Type 3d. Let *O* be the circumcentre, and draw the lines *OA*, *OB*, and *OC*. Clearly these three lines are all equal in length, being the radius of the circumscribing circle. It applis when the circumcentre, *O*, is inside thet riangle, and that happens when the triangle *ABC* is acute. The limiting case is a right angled triangle when *O* is at the mid-point of one side.



In the case of an equilateral triangle, the circumcentre, is by symmetry, also the incentre and lies on the bisector of each of the angles. Hence $\alpha = \beta = \gamma$, and so all the three smaller triangles are similar, which yet again fails to satisfy all the requirements of the problem.

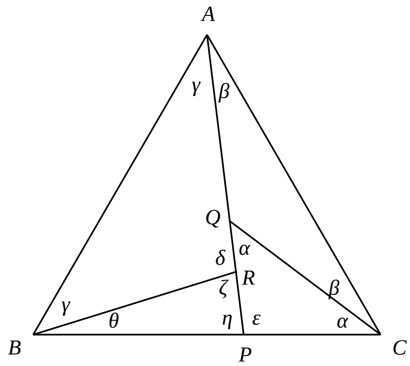
4 Pieces

There is no point in examining any dissection that does not divide all three angles of the equilateral triangle. So there will be no solution by simply adding an isosceles triangle on to the side of one of the 3-piece divisions, since the added triangle will have an undivided angle.

We cannot divide one of the triangles of Type 3d, because , it would still leave two similar triangles undivided, and that does not satisfy the problem.

Types 3a and 3b have two undivided vertices, and so cannot be used.

But we can add a cut through *B* in type 3c as shown here.



In triangle *APC*, we ry the division into two isosceles triangles with *PA* = *PC* and *QA* = *QC*. But α is the external angle of triangle *AQC*, and so $\alpha = 2\beta$. But $\alpha + \beta = C = 60^\circ$, and so we can get:

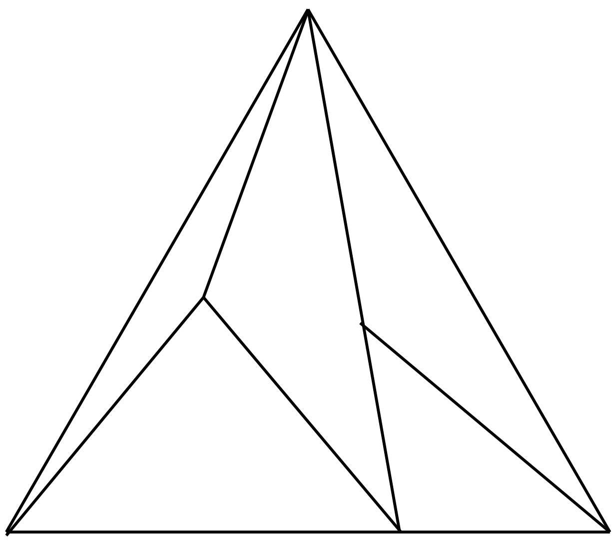
$\alpha = 40^\circ$, and $\beta = 20^\circ$.

$\beta + \gamma = A = 60^\circ$, and so $\gamma = 40^\circ$. If *ARB* is to be isosceles, and *R* lies between *A* and *P*, then we must have *RA* = *RB*, and then triangles *ARB* and *QPC* are similar, which disallows this as a solution.

But working through the other angles, since $\delta = \varepsilon$ (= 100º), which means that $\zeta = \eta$ (= 80º), and hence triangle *PBR* is also isosceles. This is an improvement on anything so far, but does not quite answer the question.

5 Pieces

Looking at the 4-piece attempt, we see that triangle *APB* is acute, and so we can divide it into three pieces using a type 3d dissection. Doing so in fact provides a valid answer to the question as posed. The base angles of the 5 isosceles triangles are 10º, 20º, 30º, 40º and 50º, which I'm sure you can verify for yourself.



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