Squares in Arithmetic Progression

In *Amusements in Mathematics*, H.E. Dudeney posed in Question 128, the problem of finding three rational squares in arithmetic progression with a common difference of 5. He gives the answer

$$\left(\frac{31}{12}, \frac{41}{12}, \frac{49}{12}\right)$$

and we can easily verify that $41^2 - 31^2 = 10\times72 = 5\times12^2 = 8\times90 = 49^2 - 41^2$.

He goes on to give an answer where the common difference is 7, and 13:

$$\left(\frac{113}{120}, \frac{337}{120}, \frac{463}{120}\right) \text{ and } \left(\frac{80929}{19380}, \frac{106921}{19380}, \frac{107729}{19380}\right)$$

Finally, he challenges the reader to find a sequence where the difference is 23, but gives no answer.

While still at school (probably around 1960), I found an answer:

$$\left(\frac{581618833}{144613560}, \frac{905141617}{144613560}, \frac{1140299183}{144613560}\right)$$

but I recorded the result only, and not whatever working I did to get there. Of course, I can no longer remember what it was, but here is one approach that could have led me to this result.

We have to find four integers, $a < b < c$, and k, such that

$$b^2 - a^2 = c^2 - b^2 = 23k^2 \text{ ....(1)}$$

From the first equality, we have $a^2 + c^2 = 2b^2$ ....(2)

Putting $m = \frac{a+c}{2}$, and $n = \frac{c-a}{2}$, we have $a = m-n$, and $c = m+n$, and (2) becomes

$$(m-n)^2+(m+n)^2 = 2m^2+2n^2 = 2b^2$$

and hence $m^2 + n^2 = b^2$

But now we have a simple Pythagorean relation. We can assume that *m* and *n* have no common factor, and so we can write:

$$m = p^2-q^2 \text{, and } n = 2pq \text{, so } b = p^2+q^2$$

Also by adding the two parts of (1) gives $2\times23k^2 = c^2-a^2 = 4mn = 2\times4pq(p^2-q^2)$

Hence *k* must be even, and so let’s write $k = 2h$.

If we want *p* and *q* to be co-prime, then each must be a square, and their difference must be divisible by 23. The sum cannot be since $23 \equiv 3 (\text{mod } 4)$ and so cannot be a factor of the sum of two co-prime squares. Putting $p = r^2$, and $q = s^2$, gives us

$$23h^2 = r^2s^2(r-s)(r+s)(r^2+s^2) \text{ ....(3)}$$

Again, r^2+s^2 must be a square, and so we have another Pythagorean situation, and we can put $r = 2tu$, and $s = t^2-u^2$, and we want one of $t^2 \pm 2tu - u^2$ to be a square, and the other to be 23 times a square.

Let's try $t = v - u$ and we consider the two factors

$$t^2+2tu-u^2 = (t+u)^2-2u^2 = v^2-2u^2 \text{ ....(4)}$$

and
$$t^2-2tu-u^2 = u^2-2uv+v^2-2u(v-u)-u^2 = v^2-4uv+2u^2 \text{ ....(5)}$$

Suppose we now try from (4) $v^2-2u^2 = w^2$, or in other words

$$(v-w)(v+w) = 2u^2$$

which looks like we should put $v=w+2$, where $v+w = u^2$, and hence $v = ½u^2+1$, and $w = ½u^2-1$. Now we can try a few low even squares to see whether anything pops out.

In fact we soon find that v = 19, w = 17, and u = 6 gives an intriguing value from (5) of -23. Perhaps we did not arrange the factors previously appropriately, and something will work out.

Going back through the substitutions gives us

$$t = v-u = 19-6 = 13$$

$$r = 2tu = 2\times13\times6 = 156$$

$$s = t^2-u^2 = 13^2-6^2 = 7\times19 = 133$$

$$r^2+s^2 = 156^2+133^2 = 205^2 = 5^2\times41^2$$

$$p = r^2 = 4^2\times3^2\times13^2 = 156^2 = 24336$$

$$q = s^2 = 7^2\times19^2 = 133^2 = 17689$$

$$m = p^2-q^2 = (r-s)(r+s)(r^2+s^2) = 23\times17^2\times5^2\times41^2 = 279340175$$

$$n = 2pq = 2\times4^2\times3^2\times13^2\times7^2\times19^2 = 860959008$$

$$b = p^2+q^2 = 24336^2+17689^2 = 905141617$$

$$a = m-n = -581618833$$

Since this is negative, it explains the fact that (5) gave a negative result. But we are interested only in the squares of numbers, so the sign does not matter.

$$c = m+n = 1140299183$$

It is interesting that all of a, b, and c are primes.

The common difference between the squares is given by

$$23k^2 = 2mn = 23\times8^2\times3^2\times5^2\times7^2\times13^2\times17^2\times19^2\times41^2 = 23\times144613560^2$$

which agrees with the answer given at the beginning.

The same technique, with u = 2, gives Dudeney's sequence with difference 7, and putting u = 4 provides an answer for a common difference of 31:

$$\left(\frac{2294239}{206640}, \frac{2566561}{206640}, \frac{2812639}{206640}\right)$$

Reference

Amusements in Mathematics, H.E Dudeney, Dover edition 1958

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