Integral Sum of Two Rational Cubes

Henry Ernest Dudeney, the English puzzle composer, asks in one of his puzzle books [1] to find two rational cubes that add to 9, other than the cubes of 1 and 2.

He elaborates only a little in saying in his answers that, as Pierre de Fermat knew, when one solution is known either as the sum or as the difference of two cubes, then it is possible to find a larger one, and claims a solution in 21 digits in numerators and denominator, although he also says that all published solutions are wrong. He then gives a smaller solution.

He further goes on to give solutions for the sum of two cubes, other than those of 3 and 8 to be 28, and another for the sum to be 6.

In a later puzzle [2] he asks for two rational cubes that add to 17.

Also, in Martin Gardner's collection of puzzles by the American Sam Loyd [3] there is a puzzle that requires to rational cubes that add to 22.

The question I asked myself is whether I could find solutions for small values, say less than 100, and also what was the method Fermat used to create another solution from a given one.

Comments

This is a problem that has exercised good mathematicians for centuries and is a difficult one. A lot is known, but here I will restrict myself to my own investigations with the aid of a computer and I am sure does not cover what is actually known for answers.

The question is equivalent to finding integral co-prime solutions for a, b and m in the equation, for given k. What small values of k are possible?

$$k m^3 = a^3 + b^3$$

We assume that $a > b$ but b could be negative, although we are looking for positive b to solve the puzzles as asked.

Fermat's technique for generating new solutions from a given one would rely on the curious identity:

$$(a^3+b^3)(a^3-b^3)^3 = a^3(a^3+2b^3)^3 - b^3(2a^3+b^3)^3 \quad \text{.................................(1)}$$

so from a positive solutions, we can find another, probably with b negative, and vice versa, but the change of sign is not guaranteed. Because we are raising each term to the ninth power on each iteration, the numbers climb to astronomical values very quickly.

Applying this for k = 9, and the starting at the cubes of 2 and 1, we get two iterations containing negative value before both terms are positive again:

$$9 \times 1^3 = 2^3 + 1^3$$

$$9 \times 7^3 = 20^3 - 17^3$$

$$9 \times 90391^3 = 188479^3 - 36520^3$$

and finally the 21-digit result Dudeney mentioned:

$$9 = \left(\frac{1243617733990094836481}{609623835676137297449}\right)^3 + \left(\frac{487267171714352336560}{609623835676137297449}\right)^3 \quad \text{..................(2)}$$

However, he said he found a smaller solution with one value negative, and from that derived this smaller positive one:

$$9 = \left(\frac{676702467503}{348671682660}\right)^3 + \left(\frac{415280564497}{348671682660}\right)^3 \quad \text{....................................(3)}$$

He must have got it from this difference of two rational cubes:

$$9 = \left(\frac{919}{438}\right)^3 - \left(\frac{271}{438}\right)^3$$

in just one iteration of the identity (1).

His solution for 28, is got by two iterations on the obvious

$$28 = 3^3 + 1^3$$

$$28 = \left(\frac{87}{26}\right)^3 - \left(\frac{55}{26}\right)^3$$

$$28 = \left(\frac{63284705}{21446828}\right)^3 + \left(\frac{28340511}{21446828}\right)^3 \quad \text{..(4)}$$

When he reported a solution for $k = 17$, he must have noticed this:

$$17 = \left(\frac{18}{7}\right)^3 - \left(\frac{1}{7}\right)^3$$

from which he derived

$$17 = \left(\frac{104940}{40831}\right)^3 + \left(\frac{11663}{40831}\right)^3 \quad \text{..(5)}$$

But Sam Loyd's result for 22, is a primitive one, and not derivable from a smaller one. It was

$$22 = \left(\frac{25469}{9954}\right)^3 + \left(\frac{17299}{9954}\right)^3 \quad \text{...(6)}$$

However, Robert Carmichael [4] gives an exercise involving this identity

$$(s^3-t^3+6s^2t+3st^2)^3 + (t^3-s^3+6t^2s+3ts^2)^3 = st\cdot(s+t)\cdot3^3\cdot(s^2+st+t^2)^3 \quad\text{...............(7)}$$

If Loyd had known this, then he could have created the puzzle based on choosing $s = 16 = 2\cdot2^3$, and $t = 11$, so $s+t = 27 = 3^3$, then the right side of (7) is 22×9954^3, and the left side is the sum of the cubes of 25469 and 17299 as given in (6).

Some results for small numbers

I give in the table below, the smallest results for each of the integers less than 100 for which I have answers. Where the smallest is via a negative solution, I report both it and the derived ones up to the positive one. In these cases, there may well be a smaller one that I have not found.

Some of them were found by means of a brute force search for sums or differences with a feasible upper limit to ensure all values were within a 64-bit word. To convert negative values to positive ones using (1) above, a multi-length calculator was employed.

Others were discovered by a search of a different type based on (7). Again, a multi-length calculator was used to convert negative values to find ma positive one.

Value	Cubes
2	$1^3 + 1^3$
6	$\left(\frac{37}{21}\right)^3 + \left(\frac{17}{21}\right)^3$
7	$2^3 - 1^3$ $\left(\frac{5}{3}\right)^3 + \left(\frac{4}{3}\right)^3$
9	$2^3 + 1^3$
12	$\left(\frac{89}{39}\right)^3 + \left(\frac{19}{39}\right)^3$
13	$\left(\frac{7}{3}\right)^3 + \left(\frac{2}{3}\right)^3$
15	$\left(\frac{683}{294}\right)^3 + \left(\frac{397}{294}\right)^3$

Value	Cubes
17	$\left(\frac{18}{7}\right)^3 - \left(\frac{1}{7}\right)^3$ $\left(\frac{11663}{40831}\right)^3 + \left(\frac{104940}{40831}\right)^3$
19	$\left(\frac{5}{2}\right)^3 + \left(\frac{3}{2}\right)^3$
20	$\left(\frac{19}{7}\right)^3 + \left(\frac{1}{7}\right)^3$
22	$\left(\frac{25469}{9954}\right)^3 + \left(\frac{17299}{9954}\right)^3$
26	$3^3 - 1^3$ $\left(\frac{75}{28}\right)^3 + \left(\frac{53}{28}\right)^3$
28	$3^3 + 1^3$
30	$\left(\frac{163}{57}\right)^3 + \left(\frac{107}{57}\right)^3$
31	$\left(\frac{137}{42}\right)^3 - \left(\frac{65}{42}\right)^3$ $\left(\frac{316425265}{119531076}\right)^3 + \left(\frac{277028111}{119531076}\right)^3$
33	$\left(\frac{1853}{582}\right)^3 + \left(\frac{523}{582}\right)^3$
34	$\left(\frac{631}{182}\right)^3 - \left(\frac{359}{182}\right)^3$ $\left(\frac{54593238059}{18048810780}\right)^3 + \left(\frac{33380537941}{18048810780}\right)^3$
35	$3^3 + 2^3$
37	$4^3 - 3^3$ $\left(\frac{19}{7}\right)^3 + \left(\frac{18}{7}\right)^3$

Value	Cubes
42	$\left(\frac{449}{129}\right)^3 - \left(\frac{71}{129}\right)^3$ $\left(\frac{12828264877}{11723102040}\right)^3 + \left(\frac{40321559123}{11723102040}\right)^3$
43	$\left(\frac{7}{2}\right)^3 + \left(\frac{1}{2}\right)^3$
49	$\left(\frac{11}{3}\right)^3 - \left(\frac{2}{3}\right)^3$ $\left(\frac{14465}{4017}\right)^3 + \left(\frac{5308}{4017}\right)^3$
50	$\left(\frac{23417}{6111}\right)^3 - \left(\frac{11267}{6111}\right)^3$ $\left(\frac{273240604181741221}{87210951352306236}\right)^3 + \left(\frac{233707858723132379}{87210951352306236}\right)^3$
51	$\left(\frac{730511}{197028}\right)^3 + \left(\frac{62641}{197028}\right)^3$
53	$\left(\frac{1872}{217}\right)^3 - \left(\frac{1819}{217}\right)^3$ $\left(\frac{12918133157903}{2729608954219}\right)^3 - \left(\frac{10253066934240}{2729608954219}\right)^3$ $\left(\frac{33154841387299518433984238326392346830569703054672960}{88264960539922401807478892670609200812800196259926 13}\right)^3 +$ $\left(\frac{50639315258668885605233901479122847978994584928 1}{88264960539922401807478892670609200812800196259926 13}\right)^3$
58	$\left(\frac{28747}{7083}\right)^3 - \left(\frac{14653}{7083}\right)^3$ $\left(\frac{650099621818168957}{190549594455179400}\right)^3 + \left(\frac{502035605557831043}{190549594455179400}\right)^3$
61	$\left(\frac{249859}{78140}\right)^3 + \left(\frac{238141}{78140}\right)^3$

Value	**Cubes**
62	$\left(\frac{11}{3}\right)^3 + \left(\frac{7}{3}\right)^3$
63	$4^3 - 1^3$ $\left(\frac{248}{65}\right)^3 + \left(\frac{127}{65}\right)^3$
65	$4^3 + 1^3$
67	$\left(\frac{5353}{1323}\right)^3 + \left(\frac{1208}{1323}\right)^3$
68	$\left(\frac{2538163}{620505}\right)^3 - \left(\frac{472663}{620505}\right)^3$ $\left(\frac{4096681218171287085661 5239}{1021173314370647718295 1970}\right)^3 + \left(\frac{1540761895176899661381 2761}{1021173314370647718295 1970}\right)^3$
69	$\left(\frac{15409}{3318}\right)^3 - \left(\frac{10441}{3318}\right)^3$ $\left(\frac{64516212068384017}{15916085197527900}\right)^3 + \left(\frac{21298792404615983}{15916085197527900}\right)^3$
70	$\left(\frac{53}{13}\right)^3 + \left(\frac{17}{13}\right)^3$
71	$\left(\frac{197}{43}\right)^3 - \left(\frac{126}{43}\right)^3$ $\left(\frac{1674586620}{414767207}\right)^3 + \left(\frac{717990337}{414767207}\right)^3$
75	$\left(\frac{17351}{3606}\right)^3 - \left(\frac{11951}{3606}\right)^3$ $\left(\frac{104456145984145201}{24991605897324612}\right)^3 + \left(\frac{31401958790537999}{24991605897324612}\right)^3$
78	$\left(\frac{5563}{1302}\right)^3 + \left(\frac{53}{1302}\right)^3$

Value	Cubes
79	$\left(\frac{13}{3}\right)^3 - \left(\frac{4}{3}\right)^3$ $\left(\frac{26897}{6783}\right)^3 + \left(\frac{17320}{6783}\right)^3$
84	$\left(\frac{433}{111}\right)^3 + \left(\frac{323}{111}\right)^3$
85	$\left(\frac{2570129}{330498}\right)^3 - \left(\frac{2404889}{330498}\right)^3$ $\left(\frac{48207549512592436837518001}{10207696238269758943360884}\right)^3 - \left(\frac{27860607119831534805838321}{10207696238269758943360884}\right)^3$ $\left(\frac{5640096041711515920130555965723671951795560946508533309441342190211570259217683849984155753750846224961}{1364346098735172709704382436820219969111828528613144916432679935956644718147531694615594911884202799208}\right)^3 +$ $\left(\frac{3315775006470417228724083504326177087086638281788962497551916247858227976193425354814087026504082975679}{1364346098735172709704382436820219969111828528613144916432679935956644718147531694615594911884202799208}\right)^3$
86	$\left(\frac{13}{3}\right)^3 + \left(\frac{5}{3}\right)^3$
87	$\left(\frac{1176498611}{216266610}\right)^3 - \left(\frac{907929611}{216266610}\right)^3$ $\left(\frac{2277504642457139640888559857318599041}{5140416356039835471013446838092418 20}\right)^3 +$ $\left(\frac{1547929841055442523385407982657129 59}{5140416356039835471013446838092418 20}\right)^3$
89	$\left(\frac{53}{13}\right)^3 + \left(\frac{36}{13}\right)^3$
90	$\left(\frac{1241}{273}\right)^3 - \left(\frac{431}{273}\right)^3$ $\left(\frac{1612982179981}{543625858776}\right)^3 + \left(\frac{2173133142899}{543625858776}\right)^3$
91	$4^3 + 3^3$

Value	Cubes
92	$\left(\frac{25903}{5733}\right)^3 - \left(\frac{3547}{5733}\right)^3$ $\left(\frac{447882708918115943}{99895475668347450}\right)^3 + \left(\frac{123135553702384057}{99895475668347450}\right)^3$
97	$\left(\frac{14}{3}\right)^3 - \left(\frac{5}{3}\right)^3$ $\left(\frac{34916}{8607}\right)^3 + \left(\frac{26815}{8607}\right)^3$
98	$5^3 - 3^3$ $\left(\frac{669}{152}\right)^3 + \left(\frac{355}{152}\right)^3$

References

[1] Dudeney, Henry E., *"The Canterbury Puzzles"*, No. 20, The Puzzle of the Doctor of Physic, Dover, 1958. Preface dated 1919.

[2] Dudeney, Henry E., *"The Canterbury Puzzles"*, No. 61, The Silver Cubes, Dover, 1958

[3] Gardner, Martin, *"Mathematical Puzzles of Sam Loyd"*, No 70, Mixed Teas, Dover 1959

[4] Carmichael, Robert D. *"Diophantine Analysis"*, Chapter 3, General exercise no.5, 1915, reprinted by Dover books 1959

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