Giving change in coins

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How many coins, on average, are needed to give change for a pound? This question is a development of [FHK] where the question was to find which combination of coin denominations gave the smallest average number used in change under the assumption that all possible values from 1 cent to 99 cents were equally likely. The number of different denominations were taken to be from 2 to 5. Some of the values given in that answer were inaccurate [AP]. This paper looks at the British equivalent, and extends the number of denominations to 6. Again, though, only change for a pound is considered.

Let's rephrase the question. What is the total number of coins required of a given set of denominations to give change once each for all of 1p to 99p, inclusive? Given our current set of coins less than £1, the following table applies.

Denominations						**Total coins required**
1						4950
1	2					2500
1	5					1150
1	10					900
1	20					1150
1	50					2500
1	2	5				1070
1	2	10				700
1	2	20				700
1	2	50				1300
1	5	10				700
1	5	20				550
1	5	50				700
1	10	20				700
1	10	50				700
1	20	50				820

Denominations						Total coins required
1	2	5	10			620
1	2	5	20			470
1	2	5	50			620
1	2	10	20			500
1	2	10	50			500
1	2	20	50			520
1	5	10	20			500
1	5	10	50			500
1	5	20	50			460
1	10	20	50			620
1	2	5	10	20		420
1	2	5	10	50		420
1	2	5	20	50		380
1	2	10	20	50		420
1	5	10	20	50		420
1	2	5	10	20	50	340

If we had to remove just one denomination from the coinage, it would save more coins by removing the 10p piece; and if we had only two moulds to make coins, then use them for 1p and 10p.

If we could choose any denominations, then the best set of denominations for saving the total number required would be as shown in the following table.

To make change for less than £1 with no more than 6 coins, the following table applies.

Denominations						Total coins required
1	10					900
1	11					900
1	12	19				515
1	5	18	25			389

Denominations						Total coins required
1	5	18	29			389
1	5	16	23	33		329
1	4	6	21	30	37	292
1	5	8	20	31	33	292

It is safe to say that these combinations would never be implemented, but if they were, it would do wonders for mental arithmetic.

As an example of the way change can be made, the following table covers the two best solutions for 6 coins.

change	*37*	*30*	*21*	*6*	*4*	*1*	*coins*
0							0
1						1	1
2						2	2
3						3	3
4					1		1
5					1	1	2
6				1			1
7				1		1	2
8					2		2
9					2	1	3
10				1	1		2
11				1	1	1	3
12				2			2
13				2		1	3
14				1	2		3
15				1	2	1	4
16				2	1		3
17				2	1	1	4
18				3			3
19				3		1	4
20				2	2		4

change	*33*	*31*	*20*	*8*	*5*	*1*	*coins*
0							0
1						1	1
2						2	2
3						3	3
4						4	4
5					1		1
6					1	1	2
7					1	2	3
8				1			1
9				1		1	2
10					2		2
11					2	1	3
12					2	2	4
13				1	1		2
14				1	1	1	3
15					3		3
16				2			2
17				2		1	3
18				1	2		3
19				1	2	1	4
20			1				1

change	*37*	*30*	*21*	*6*	*4*	*1*	*coins*
21			1				1
22			1			1	2
23			1			2	3
24			1			3	4
25			1		1		2
26			1		1	1	3
27			1	1			2
28			1	1		1	3
29			1		2		3
30		1					1
31		1				1	2
32		1				2	3
33			1	2			3
34		1			1		2
35		1			1	1	3
36		1		1			2
37	1						1
38	1					1	2
39	1					2	3
40		1		1	1		3
41	1				1		2
42			2				2
43	1			1			2
44	1			1		1	3
45	1				2		3
46			2		1		3
47	1			1	1		3
48			2	1			3
49	1			2			3
50	1			2		1	4
51		1	1				2
52		1	1			1	3
53	1			2	1		4
54			2	2			4

change	*33*	*31*	*20*	*8*	*5*	*1*	*coins*
21			1			1	2
22			1			2	3
23			1			3	4
24				3			3
25			1		1		2
26			1		1	1	3
27			1		1	2	4
28			1	1			2
29			1	1		1	3
30			1		2		3
31		1					1
32		1				1	2
33	1						1
34	1					1	2
35	1					2	3
36		1			1		2
37		1			1	1	3
38	1				1		2
39		1		1			2
40			2				2
41	1			1			2
42	1			1		1	3
43	1				2		3
44		1		1	1		3
45			2		1		3
46	1			1	1		3
47		1		2			3
48			2	1			3
49	1			2			3
50	1			2		1	4
51		1	1				2
52		1	1			1	3
53	1		1				2
54	1		1			1	3

change	*37*	*30*	*21*	*6*	*4*	*1*	*coins*
55		1	1		1		3
56		1	1		1	1	4
57		1	1	1			3
58	1		1				2
59	1		1			1	3
60		2					2
61		2				1	3
62	1		1		1		3
63			3				3
64	1		1	1			3
65	1		1	1		1	4
66		2		1			3
67	1	1					2
68	1	1				1	3
69	1	1				2	4
70	1		1	2			4
71	1	1			1		3
72		1	2				3
73	1	1		1			3
74	2						2
75	2					1	3
76	2					2	4
77	1	1		1	1		4
78	2				1		3
79	1		2				3
80	2			1			3
81		2	1				3
82	2				2		4
83	1		2		1		4
84	2			1	1		4
85	1		2	1			4
86	2			2			4
87		2	1	1			4
88	1	1	1				3

change	*33*	*31*	*20*	*8*	*5*	*1*	*coins*
55	1		1			2	4
56		1	1		1		3
57	1			3			4
58	1		1		1		3
59		1	1	1			3
60			3				3
61	1		1	1			3
62		2					2
63		2				1	3
64	1	1					2
65	1	1				1	3
66	2						2
67	2					1	3
68	2					2	4
69	1	1			1		3
70		2		1			3
71	2				1		3
72	1	1		1			3
73	1		2				3
74	2			1			3
75	2			1		1	4
76	2				2		4
77	1	1		1	1		4
78	1		2		1		4
79	2			1	1		4
80	1	1		2			4
81	1		2	1			4
82		2	1				3
83		2	1			1	4
84	1	1	1				3
85	1	1	1			1	4
86	2		1				3
87	2		1			1	4
88	2		1			2	5

change	*37*	*30*	*21*	*6*	*4*	*1*	*coins*
89	1	1	1			1	4
90		3					3
91		3				1	4
92	1	1	1		1		4
93		1	3				4
94	1	1	1	1			4
95	2		1				3
96	2		1			1	4
97	1	2					3
98	1	2				1	4
99	2		1		1		4

change	*33*	*31*	*20*	*8*	*5*	*1*	*coins*
89	1	1	1		1		4
90		2	1	1			4
91	2		1		1		4
92	1	1	1	1			4
93		3					3
94	2		1	1			4
95	1	2					3
96	1	2				1	4
97	2	1					3
98	2	1				1	4
99	3						3

The problem seems very resistant to analysis, requiring a full search of the possible combinations to make sure that the optimum is obtained.

References

[AP] Andy Pepperdine, Commentary on problem 1773, *Journal of Recreational Mathematics*, (23:4, p 307) 1991

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[FHK] Friend H. Kierstead, Jr., Solution to problem 1773, *Journal of Recreational Mathematics*, (23:1, p 71) 1991

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