A sequence of points in disjoint intervals

Andy Pepperdine

Consider a sequence of numbers, each lying between 0 and 1. Further, the first two are each in separate halves of the interval [0, 1]; the first three are in separate thirds of the whole interval, the first four are in separate quarters, etc. Is there a limit to how many terms there can be in such a sequence?

I first came across this problem in [2], where a solution is given for 14 points, but there is a footnote on page 62 that says, "M. Warmus has proved quite recently the number n = 17 to be the last one for which the problem has a solution." There is also a page on Mathworld [3] giving further references.

Commentary

Surprisingly, the answer is Yes. A maximum of 17 points can be selected to conform to the conditions.

It makes an interesting simple problem in interval arithmetic to program a computer to find one, and even to count how many solutions there are. For the problem to make sense, it is necessary to replace points by intervals and to say that two sequences are equivalent if corresponding points in each sequence are in the same interval. I used it for some time as a benchmark for testing compiler optimisations, but as processor speeds have increased, it now takes only a very short time to find all solutions.

Solution

There are a total of 1536 solutions, but they occur in pairs since if, for i = 1,17 $\{a_i\}$ is a solution, then so is $\{1-a_i\}$. One set of 17 integers in the range [1, 100] that satisfy the conditions (suitably scaled) is:

05 54 85 27 71 42 99 17 62 35 78 11 49 92 23 68 41

One interval solution that has the largest range of its smallest interval is:

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( 0/ 1, 1/17)  ( 7/13, 6/11)  (11/13, 6/ 7)  ( 4/15, 3/11)
(12/17, 5/ 7)  ( 5/12, 3/ 7)  (16/17, 1/ 1)  ( 2/15, 1/ 7)
( 8/13, 5/ 8)  ( 1/ 3, 6/17)  (10/13,11/14)  ( 1/ 5, 3/14)
( 8/17, 1/ 2)  (15/17,13/14)  ( 1/15, 2/17)  (11/17,11/16)
                        ( 6/17, 7/17)
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The smallest interval is 1/165 long, or about 0.006061.

References

[1] Creative Commons License, *http://creativecommons.org/licenses/by/2.5/*

[2] Steinhaus, Hugo, *One Hundred Problems in Elementary Mathematics*, Pergamon Press, 1963 (translated from *Sto zadań*, PWN Warszawa, 1958)

[3] Weisstein, Eric W. *"18-Point Problem."*, http://mathworld.wolfram.com/18-PointProblem.html

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