Topswops

This is an updated version of an article that appeared in *The Mathematical Gazette*, 73, June 1989, p.131.

In his book *Time Travel and other mathematical bewilderments* Martin Gardner describes a series of shuffles ascribed to J.H. Conway. One in particular called *topswops*. In this shuffle, a pack of cards is taken where only the numbers matter. They are held face up, and the top number is noted; call it *M*. Then the top *M* cards are counted off (thus reversing the order) and replaced onto the deck again. Then whichever is the topmost card is again noted and the corresponding number of cards counted off. Here we will restrict the discussion to a pack of *N* cards with the values 1, …, *N*.

Clearly, when the top card is an ace, then the shuffle stops and there is a loop of one card long. It can easily be shown, as iin Martin Gardner's book, that any permutation of of a set of cards from ace upwards, must always end with this degenerate loop. But how long can the sequence be before that time arrives? It was not known what that value was for a single suit of 13 ordinary playing cards.

I set out to answer this with (at the time) a fairly powerful computer. Consider the sequence of permutations

$$s_0, s_1, \ldots s_i, s_{i+1}, \ldots$$

where s_{i+1} is derived from s_i, by means of a topswops shuffle. Let s_M be the first in the sequence for which the top card is an ace. Then all $s_j, j > M$, also have the top card an ace, and we are interested in the values of *M* as a function of the number of cards, *N*.

The brute force search technique would examine all the $N!$ permutations for s_0 and try them in turn to find the maximum value of the corresponding *M*. But we can do better than that by considering the sequence from s_M to s_0. Regress can only be made from s_j to s_{j-1} when the *k*-th card in the pack for at least one *k*. If this is not the case for any $k > 1$, then we have reached s_0. The question is now rephrased: what is the longest such sequence of permutations each of which always has at least one card in its corresponding ordinal position? This card is moved to the top in the reverse shuffle to get the previous permutation in the sequence.

Thus starting with a given s_M as the root, we can construct a tree of permutations, such that each node has for its children the possible predecessors in the topswops sequence. All we need to do is write a computer program to build and search this tree. Since all we are interested in is the total distance from root to leaf, this is quite feasible – given enough time.

We already know that the top card of s_M is an ace. So the possible roots of the trees would be the $(N-1)!$ permutations of the numbers 2,…,*N*, with a 1 prepended. We can do even better by introducing jokers into the pack. At s_M let us put jokers in positions 2,…,*N*, position 1 being the ace. Now we can build the tree from a single root where the children of a node are derived from it by the rule

> If at a node the *k*-th ($k > 1$) position is occupied by a joker, and *k* does not appear, then replace the joker by *k*.
>
> Then, if the *k*-th position is *k*, apply the reverse topswops shuffle.

Some of the leaves in this tree will still have jokers in the pack, but in positions whose ordinals are found elsewhere, for example

(6 joker joker 2 1 3)

In this way the question was answered for all numbers *N* from 1 to 13 over a long weekend. These results are as follows

N	Length	Permutations giving maximum length
1	0	1
2	1	2 1
3	2	3 1 2 2 3 1
4	4	2 4 1 3 3 1 4 2
5	7	3 1 4 5 2
6	10	4 5 6 2 1 3 5 6 4 1 3 2 3 6 5 1 4 2 4 1 5 2 6 3 4 1 6 5 2 3 (*)
7	16	3 1 4 6 7 5 2 4 7 6 2 1 5 3
8	22	6 1 5 7 8 3 2 4
9	30	6 1 5 9 7 2 8 3 4
10	38	5 9 1 8 6 2 10 4 7 3
11	51	4 9 11 6 10 7 8 2 1 3 5
12	65	2 6 1 10 11 8 12 3 4 7 9 5 (*)
13	80	2 9 4 5 11 12 10 1 8 13 3 6 7

I was surprised that there are so few sequences with the maximum length. That would be an area for further research. The other remarkable thing is that those marked with a (*) do not end up in the natural order after the maximum length shuffle. The 6 case ends with (1 4 3 2 5 6) with the 5 the last card to be moved; whereas the 12 case gives (1 6 5 2 3 4 7 8 9 10 11 12) with 7 being the last card to be moved.

The computer search was performed in Ada on a ND 5700 cpu.

Addendum

Of course with the march of technology, a re-run of the program, converted to C, took less than 2 minutes on a standard desktop PC (April 2019).

The sequence of lengths has an entry at the Encyclopedia of Integer Sequences https://oeis.org/A000375 where the next few values are: 14 – 101; 15 – 113; 16 – 139; 17 – 159.

Andy Pepperdine